Meaning and Discontinuity in Consumer Choice*

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Abstract

Continuity is a basic assumption in consumer theory, and it seems rather plausible when physiological processes are taken into account. But when consumption is the carrier of meaning, discontinuities may arise. Specifically, consumer preferences may behave discontinuously at zero quantities, as in the case of vegetarians who prefer not to consume any amount of animal meat. We argue that, as opposed to the example of lexicographic preferences, the discontinuity in such cases is not only in stated preferences, but can also be matched by consumption behavior. Relatedly, it can be represented by a numerical utility, and we provide an axiomatization of such a function.

1 Introduction

Individuals, households, and organizations often state facts about their preferences. Verbal and textual statements may be used for communicating preferences for the sake of coordination, as well as for declaring or justifying policies

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and regulations. However, not all such statements are equally credible. Some descriptions of preferences may sound more convincing than others and some policies are more likely to be implemented than others. In particular, natural language may easily describe discontinuous preferences, and these may raise the economist's eyebrow. Consider the following examples.

Example 1: A consumer declares she is vegetarian. It is a matter of principle for her not to consume animals' meat. The amount of meat used in a product is immaterial to her; the very fact that it exists at a positive level is distinctly different from non-existence. Obviously, she cannot tell whether a dish contains a minuscule amount of meat, but she attaches meaning to the act of consumption, and meaning introduces discontinuity at zero quantity.

Example 2: John writes his will, leaving his estate to his two children. He would like each of the children to have as much property as possible, but it is also important to him to have an equal division between them. This requires selling some assets, incurring transaction costs. John tells his lawyer that, as long as the costs do not exceed 5% of the estate, he prefers to incur them for the sake of an equal division.

Example 3: The president of a university expresses preferences for faculty diversity, but, she says, diversity will not come at the expense of academic quality. The stated preferences are lexicographic: academic quality is a primary goal, and only in case of indifference does the secondary principle have an effect. As pointed out by Rubinstein (1998), when preferences are defined in a language, rather than represented by a mathematical function, lexicographic preferences might be rather natural.

These examples involve agents who wish to abide by a certain principle (vegetarianism, fairness, faculty diversity, respectively) and their choices are expected to embody that principle. As a result, their preferences are dictated by the *meaning* carried by the choice objects, and not only by their physical characteristics. However, there are subtle differences between the examples and we maintain that not all three stand on an equal footing.

In Example 1 discontinuity of preferences only occurs along the subspace of bundles that contain zero quantity of meat. Relatedly, such preferences may admit a numerical representation. For example, assume that $u(x)$ measures the consumer's hedonic utility, and that she maximizes the function

$$U(x) = \begin{cases} u(x) & x \text{ is vegetarian} \\ u(x) - \gamma & \text{otherwise} \end{cases} \tag{1}$$

where $\gamma > 0$ measures the degree to which she cares about vegetarianism. Specifically, if $u(x)$ is bounded from above but not from below, maximization of U can capture the behavior of a vegetarian who would consume meat rather than die of hunger, but who would not do so for the sake of sheer pleasure.

Similarly, John's preferences in Example 2 also exhibit discontinuity along a specific subspace: if we denote the two children's shares of the estate by $(y_1, y_2) \in [0,1]^2$, John assigns a special value to the subspace $y_1 = y_2$. We could capture his stated preferences by the function

$$U(y_1, y_2) = \begin{cases} u(y_1, y_2) & y_1 = y_2 \\ u(y_1, y_2) - \gamma & \text{otherwise} \end{cases} \tag{2}$$

where $u(y_1, y_2)$ is a symmetric function and $\gamma > 0$ is the value put on equality.

By contrast, it is well-known (Debreu, 1954, 1959) that the lexicographic preferences of Example 3 are nowhere continuous and do not admit a representation by any numerical utility function. One way to conceptualize the difference between Example 3 and Examples 1,2 is that in the latter meaning is attached only to a subspace of the space of alternatives. In Example 1, the meaning "this is a vegetarian dish" is attached to the subspace in which the quantities of all non-vegetarian products are zero. In Example 2, meaning is attached to the diagonal, to be read as "the two children were treated symmetrically". By contrast, in Example 3 meaning is attached to any conceivable level of "academic quality". One way to interpret the non-existence of a numerical representation of preferences in Example 3 is that the real line is rich enough to allow for a meaningful distinction, but not for uncountably many such distinctions.

These differences might suggest that the stated preferences in Examples 1,2 are more convincing than those in Example 3. Indeed, many consumers are vegetarian and seem to be willing to give up hedonic well-being for the principle they espouse, in a way that corresponds to maximization of U. Along similar lines, it stands to reason that John might wish his will to be executed as stated. But when it comes to Example 3, economists might wonder whether actual behavior would follow the stated lexicographic preferences. If "academic quality" is perceived as a continuous variable, the statement made might be in conflict with actual decisions (if not vacuous).

However, the fact that some economists find the stated preference in Examples 1,2 more convincing than those in Example 3 may be a cultural matter. Economists are used to model preferences by a utility function, and their intuition regarding the veracity of stated preferences might well be colored by the existence of a numerical representation of these preferences. But this criterion might be questioned by others. For example, psychologists might argue that such a numerical representation does not correspond to any known mental process and is therefore an odd choice for a test of statements about preferences. On the other hand, mathematicians might point out that economists tend to choose representations in the real line due to convenience alone, and that more general structures, such as non-standard analysis, might be equally acceptable as representations of preferences.[1]

Rather than assuming that a certain mathematical structure is a natural choice for representing preferences, we take here an axiomatic approach. The goal of the axiomatization is to translate statements about preferences to the language of concrete choices, in the hope that such a translation would make it easier for listeners to judge how convincing a statement is. We provide an axiomatic derivation of consumer preferences that can be described by a function U as in (1,2). Characterizing the binary relations that admit such a representation allows us to judge the plausibility of the verbal statement captured by the function. Moreover, such an exercise can help us determine which functional form is appropriate to use in order to model the discontinuity introduced by vegetarianism or egalitarianism. Observe that our focus here is on the mind-experiments that are called for in order to evaluate natural-

[1] Indeed, Blume, Brandenburger, and Dekel (1991) use lexicographic structures, arguing that they can best capture reasoning about equilibrium refinements.

language statements, and in order to judge specific functional forms, rather than on empirical tests of the axioms involved. That is, we do not study axioms under the assumption that only revealed preferences are available as data. Rather, we use axioms to compare conceivably-revealed preferences to (observable) linguistic descriptions of preferences.

Clearly, the utility function axiomatized here is rather special in two ways. First, it deals with a single source of meaning. That is, it can describe the behavior of the vegetarian consumer, but should the latter also care about child labor or Fair Trade, we will have to consider more general functional forms, allowing for discontinuities at several subspaces. Second, the meaning attached to consumption is dichotomous. For example, if our consumer prefers not to eat animals at all, but, should she have to, prefers to eat seafood than mammals, we will again find that (1) is too special to describe her preferences. For both reasons one might be interested in a functional form with several additive terms such as γ above. Importantly, the resulting model would still involve discontinuities in quantities. Similarly, people who keep kosher can vary greatly in terms of what consumption they consider to be allowed by the principle, but these distinctions tend to be qualitative rather than quantitative.[2]

The axiomatization of preferences as in (1) is provided in Section 2. It turns out that (with only one source of meaning), little needs to be assumed to obtain this additively separable representation. A survey of related literature is provided in Section 3. Section 4 concludes with a general discussion.

[2]For example, there are Jews who do not keep kosher in general, but refrain from eating pork, which has become a symbol of non-kosher food. Others would not bring non-kosher food home, but would eat anything they like away from home. Yet, casual observation suggests that very few would say, "I keep kosher, and do not eat more than 100g of pork a week."

2 Axiomatization

2.1 Set-up

The alternatives are consumption bundles in X, which is a closed and convex subset of $\mathbb{R}^n_+$. For each good $i \leq n$ there is an indicator $d_i \in \{0,1\}$ denoting whether the good violates the principle. That is, $d_i = 1$ implies that the good is inconsistent with the principle (say, contains meat), and $d_i = 0$ – that it doesn't (purely vegetarian). The consumer is aware of the vector $d \in \{0,1\}^n$, where we assume that producers should and do truthfully disclose the ingredients of their products.[3]

We wish to axiomatize the model in which, given d, the consumer maximizes $U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}}$where $d \cdot x$ is the inner product of the two vectors, so that $d \cdot x > 0$ if and only if there exists a product i that violates the principle ($d_i = 1$) and that is consumed at a positive quantity in x.

In this section we assume that the vector d is known and kept fixed. That is, the consumer is provided with information about the goods that are and are not vegetarian, and we implicitly assume that this information is trusted. We keep the information fixed, and can therefore suppress d from the notation, assuming that a binary relation $\succsim_d = \succsim \subset X \times X$ is observable. The information contained in the vector d is summarized by the answer to the question, is $d \cdot x > 0$? We thus define $X^0 = \{x \in X \mid d \cdot x = 0\}$, that is, all consumption bundles that do not use any positive amount of the "forbidden" goods, while $X^1 = X \backslash X^0 = \{x \in X \mid d \cdot x > 0\}$ contains the other bundles. Observe that X^0 is closed and convex and X^1 is convex.

Before moving on, we introduce some notation. The term "a sequence $(x_n)_{n \geq 1} \to_{n \to \infty} x$" will refer to a sequence $(x_n)_{n \geq 1}$ such that $x_n \in X$ for all n, and $x_n \to_{n \to \infty} x$ in the standard topology, where $x \in X$. When no ambiguity is involved, we will omit the index notation "$n \to \infty$" as well as the subscript "$n \geq 1$". We will use the notation "a sequence $(x_n) \subset A$" for "a sequence

[3]Example 2 can fit into this set-up by changing the variables, exploiting the symmetry assumption. Specifically, we can set $x_1 = \max(y_1, y_2)$ and $x_2 = \max(y_1, y_2) - \min(y_1, y_2)$ with $d = (0,1)$.

$(x_n)_{n\geq 1}$ such that $\{x_n\}_{n\geq 1} \subset A$". Conditions that involve an unspecified index such as $x_n \gtrsim y_n$ are understood to use a universal quantifier ("for all $n \geq 1$"). Finally, when no confusion is likely to arise we will also omit the parentheses and use $x_n \to x$ rather than $(x_n) \to x$.

2.2 Axioms

We impose the following axioms on $\gtrsim$. We start with the standard assumption positing that choice behavior is described by a complete preorder.

A1. Weak Order: $\gtrsim$ is complete and transitive on X.

The next axioms will make use of the following key notion:

Definition 1 *Two sequences $x_n \to x$ and $y_n \to y$ are* comparable *if*
(A) there exist $i, j \in \{0,1\}$ such that $(x_n) \subset X^i, x \in X^i$ and $(y_n) \subset X^j, y \in X^j$
or
(B) there exist $i, j \in \{0,1\}$ such that $(x_n), (y_n) \subset X^i$ and $x, y \in X^j$.

Clearly, if all of the elements of $(x_n), (y_n)$, as well as the limit point of each are in the same subspace – X^0 or X^1 – the sequences are comparable.[4] However, two sequences $x_n \to x$ and $y_n \to y$ are comparable also in two other cases: first, (A) if (x_n) as well as *its* limit x are all in one subspace, while (y_n) with *its* limit, y, are all in another. And, second, (B) if the elements of both sequences belong to X^1 and the limits of both belong to X^0. (In principle, the opposite is also allowed by the definition, but X^0 is closed, so we cannot have a sequence in it converging to a point in X^1.) Basically, comparability rules out cases in which the transition to the limit makes only one sequence cross the boundary between the subspaces, leaving X^1 and reaching X^0. If this occurs, then the information we gather from preferences along the sequences is not very useful for making inferences about the limits: one sequence changes in a way that is discontinuous, and the other one doesn't. (See Fig. 1.)

By contrast, if the two sequences are comparable because none of them crosses the boundary between the two subspaces, then there is no reason for

[4] Here and in the sequel we use the terms "space" and "subspace" in the topological sense.

any violation of continuity. And, importantly, if both do cross the boundary, we still expect preference information along the sequences (where both (x_n) and (y_n) are in one subspace, which can only be X^1 in this case) to carry over to the limits (even though these are located in another subspace).

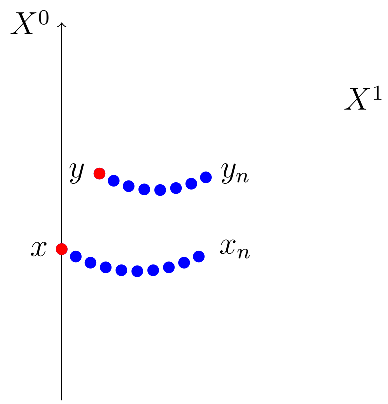

Figure 1: The comparability notion rules out this case.

We can now state our continuity axiom:

A2. Weak Preference Continuity: For all comparable sequences $x_n \to x$ and $y_n \to y$, if $x_n \succsim y_n$ for all n, then $x \succsim y$.

Observe that, without the comparability condition, A2 would be a standard, though rather strong axiom of continuity: it would simply say that the graph of the relation $\succsim$ is closed in $X \times X$. This axiom is stronger than the standard continuity axiom of consumer choice, though it is implied by it when the relation $\succsim$ is also known to be a weak order. In our case, however, the consequent of the axiom is only required to hold if the sequences are comparable. As explained above, $x_n \succsim y_n$ for all n may not imply $x \succsim y$ (in the limit) if, for example, y is the only element involved that is in X^0; in this case it can enjoy the extra utility derived from obeying the principle, and thus $y \succ x$ can occur at the limit with no hint of this preference emerging along the sequence.

Clearly, if we restrict attention to one subspace, that is, if all of (x_n), (y_n), x, y are in X^1 or if all of them are in X^0, we obtain a standard continuity condition. Indeed, this would suffice to represent $\succsim$ on X^0 by a continuous utility function u^0 and to represent it on X^1 by a continuous utility function u^1, where u^0 and u^1 (having disjoint domains) need not have anything in common.

While A2 deals with weak preferences that are carried over to the limit, we will also need a corresponding axiom for strict preferences:

A3. Strict Preference Continuity: For all comparable sequences $x_n \to x$ and $y_n \to y$, and all $z, w \in X$, if $x_n \succsim z \succ w \succsim y_n$ for all n, then $x \succ y$.

To see the meaning of this axiom, assume, again, that comparability were not required. In this case, $x_n \succsim z$ and $w \succsim y_n$ would imply $x \succsim z$ and $w \succsim y$, respectively, and from $z \succ w$ we would easily conclude $x \succ y$. In our case, however, we could have that $(x_n) \subset X^1$ and $x \in X^0$, and thus we cannot conclude that $x \succsim z$ (and, naturally, the same holds for w and y). Yet, comparability of $x_n \to x$ and $y_n \to y$ suffices to conclude that the preference gap between z and w is indeed enough to guarantee a strict preference between x and y.

Next, we introduce an Archimedean axiom stating that the "cost" of the principle in terms of utility is strictly positive, and, moreover, that no utility difference over X^0 exceeds infinitely many such "costs". Specifically, consider a sequence $(z_n) \subset X^1$ that converges to a point $z \in X^0$. In terms of hedonic utility, the bundles z_n become practically indistinguishable from z. However, the fact that z satisfies the principle means that its overall utility is higher than the limit of the corresponding utility values along the sequence. Intuitively, reaching X^0 at the limit provides an extra utility boost, which is not captured by the (continuous) hedonic utility, but should be captured in our overall-utility representation. One way to see this in terms of preferences is the following: if, along the sequence, $z_n \sim y \in X^0$, then we should have strict preference at the limit, $z \succ y$. In this case, the (hedonic) utility gap between z and y is a measure of the contribution of the principle to overall utility. The axiom states that, when aggregated, these measures are large enough to cover the entire utility range over X^0.

That is, an infinite chain of such preference gaps in X^0—as shown in Fig. 2, $x^k \succ x^{k+1} \succ x^{k+2} \succ \ldots$— cannot be bounded above or below by some bundle $\hat{x}$ (otherwise $\hat{x}$ would take an infinite utility value). We thus ensure that the utility value of any bundle in X^0 be finite, and that the (hedonic) utility difference between any two alternatives in X^0 can always be measured in terms

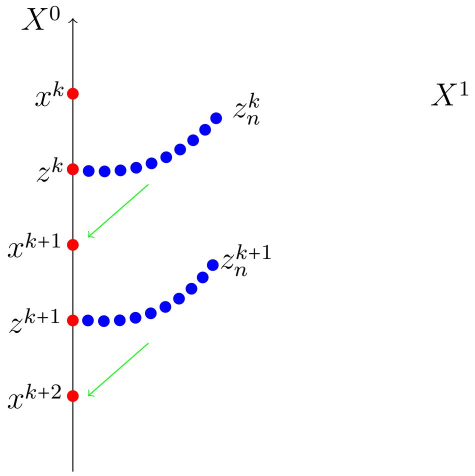

Figure 2: An illustration of A4

of finitely many "costs" of the principle.

Explicitly,

A4 Archimedeanity: Let $(x^k, z^k) \subset X^0$ and $(z_n^k)_{n,k\geq1} \subset X^1$ be such that (i) $z_n^k \to z^k$, (ii) $x^k \gtrsim z^k$ and (iii) $z_n^k \gtrsim x^{k+1}$ for all $k \geq 1$ ($z_n^k \gtrsim x^{k-1}$ for $k \geq 2$) and for all $n \geq 1$. Then there does not exist $\hat{x} \in X$ such that $x^k \gtrsim \hat{x}$ ($\hat{x} \gtrsim x^k$) for all $k \geq 1$.

Finally, we find it convenient to rule out the case in which all points in X^0 are equivalent.

A5 Non-Triviality: There are $x, y \in X^0$ such that $x \succ y$.

2.3 Results

We are now ready to state our behavioral characterization of preferences that satisfy the aforementioned axioms.

Theorem 1 *Let there be given $d \in \mathcal{D}$ and $\gtrsim$. The relation $\gtrsim$ satisfies A1-A5 if and only if there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a constant $\gamma > 0$ such that $\gtrsim$ is represented by*

$$U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}} \tag{3}$$

Representation (3) captures an agent whose choices are driven by two factors: on the one hand, the desire to maximize hedonic well-being – measured, as usual, by u – and, on the other hand, the desire to abide by an intrinsic principle – whose violation affects overall well-being by the penalty γ. We note that axioms A1-A5 do not seem to explicitly demand that hedonic well-being and value utility be additively separable. As explained shortly, this is basically a result of the ordinality of the utility function in standard consumer theory. In bold strokes, we use the cost of violating the principle as a unit of measurement by which the hedonic utility can be scaled.

The proof of Theorem 1 appears in the Appendix. To better understand its logic, we first note that axioms A1 and A2 trivially imply that one can find continuous representations of $\succsim$ on X^0 and on X^1, because on each of these A2 implies the standard continuity axiom. This, however, does not mean that there exists a function that is continuous on all of X and that represents $\succsim$ both on X^0 and on X^1 (separately). The Online Appendix is devoted to an auxiliary result, stating that A3 is the missing link. The result (formally stated in the online appendix) says that any bounded and continuous utility function that represents $\succsim$ on X^1 has a unique continuous extension to X^0, in such a way that the extension represents $\succsim$ also on X^0. Thus, A1-A3 can help us find functions that we can think of as the hedonic utility u above: each is continuous throughout X and correctly represents preferences on each of X^0, X^1. We do not expect any of them to represent preferences across the two spaces, because we know that discontinuities are to be observed between them. More concretely, when a sequence in X^1 converges to a limit point in X^0, we expect the overall utility, U, to "jump" in a discontinuous way, where the utility of limit point gets the boost of obeying the principle (while having practically the same hedonic utility u as the tail of the sequence).

The question we turn to is whether, among all the functions u as above, there exist some for which the "boost" in utility when reaching X^0 is a constant $\gamma > 0$. Axiom A4 guarantees that this is possible. First, it guarantees that the boost is strictly positive at any point in X^0. Indeed, axioms A1-A3 do not preclude the possibility that at some points in X^0 the principle is valued (corresponding to $\gamma > 0$) while at others it isn't (as if $\gamma = 0$). A4 does preclude this possibility. More formally, it is easy to see that A4 implies

Discontinuity: Let $x, y \in X^0$, and let there be a sequence $x_n \to x$ with $(x_n) \subset X^1$ such that $x_n \succsim y$. Then $x \succ y$.

As explained in the presentation of A4, we may focus on the case $x_n \sim y$, where standard continuity (over all of X) would imply $x \sim y$, whereas the Discontinuity condition *demands* that a utility gain will be obtained at the limit, to result in strict preference $x \succ y$. Once this is guaranteed, to obtain a representation as in (3) one may exploit the ordinality of standard utility representations and choose to "scale" the utility in such a way that the boost is indeed a constant $\gamma > 0$. This is basically the strategy of the proof: we define "steps" on X^0 that intuitively correspond to "better than... by exactly the utility of the principle", find a utility function that increases by the same amount for each such step, and extend it to all of X. For this strategy to succeed, we also need to make sure that these steps go far enough. In other words, we wish to make sure that no alternative in X^0 is so much better than any other that the utility difference between them is incommensurable with the steps. And this is the precise meaning of A4.

While A4 is somewhat cumbersome, it is implied by the following axiom:
A6 Lipschitz: There exists $\delta > 0$ such that, for every $x, y, z \in X^0$, and every sequence $z_n \to z$ with $(z_n) \subset X^1$ such that $x \succsim z$ and $z_n \succsim y$, we have $\|x - y\| > \delta$.

Axiom A6 states that, for a bundle $x \in X^0$ to be better than another bundle, $y \in X^0$, by "at least the cost of the principle", x should not be too close to y. We dub it "Lipschitz" as it will be satisfied by any utility function that is Lipschitz continuous on the entire space. Observe, however, that we only require the Lipschitz condition for one specific $\delta > 0$, guaranteeing that two bundles that are δ-close will not have a utility gap that is higher than a certain threshold (the presumed γ). If we restrict attention to compact bundle spaces, we can use A6 in lieu of A4:

Corollary 1 *Let there be given $d \in \mathcal{D}$ and $\succsim$ and assume that X is compact. If the relation $\succsim$ satisfies A1-A3, A5, and A6, there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a constant $\gamma > 0$ such that $\succsim$ is represented by $U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}}$*

The Appendix also contains the brief proof of this result.

To what extent is the representation unique? The answer depends on the range of u and on γ. For example, if $\gamma > \sup_{x \in X^1} (u(x)) - \inf_{x \in X^0} (u(x))$, we have $U(x) > U(y)$ for all $x \in X^0, y \in X^1$ and the consumer would never give up the principle. In this case the utility function is only ordinal: any monotone transformation of u and γ that satisfies the above inequality represents preferences, and the utility function is far from unique. If, by contrast, γ is very small relative to $\sup_{x \in X^1} (u(x)) - \inf_{x \in X^0} (u(x)) > 0$, the monotone transformations that respect the representation (3) are much more limited. As will be clear from the proof, one can choose u more or less freely until a point of equivalence between two bundles $x \in X^0, y \in X^1$, and then the utility is uniquely determined throughout the preference-overlap between X^0 and X^1. Clearly, shifting u by a constant and multiplying both u and γ by a positive constant is always possible. Thus, on the preference-overlap between X^0 and X^1 we have a cardinal representation, and outside this preference interval – only an ordinal one.[5]

3 Related Literature

It has long been observed that consumers care about ethical values. Auger, Burke, Devinney, and Louviere (2003) and Prasad, Kimeldorf, Meyer, and Robinson (2004) find that consumers are conscientious and express willingness to pay more for products that have desirable social features, such as environmental protectionism, avoiding child labor, as well as sweatshops. Barnett, Cloke, Clarke, and Malpass (2005) discuss the notion of "consuming ethics". De Pelsmacker, Driesen, and Rayp (2005) estimate the willingness to pay for coffee with and without the label "Fair Trade" and find significant differences.[6] Similar results are found in Loureiro and Lotade (2005), Basu and Hicks (2008), and Hainmueller, Hiscox, and Sequeira (2015) who show

[5]This is reminiscent of the degree of uniqueness of representations of a semi-order by a function u and a just-noticeable-difference $\delta > 0$. See, for instance, Beja and Gilboa (1992).

[6]About 10% of the sampled consumers are willing to pay a premium that is 27% for the label.

that the "Fair Trade" label increase sales by 10%. Enax, Krapp, Piehl, and Weber (2015) report neurological evidence for the positive effects of social sustainability. The standard methodology in these studies is discrete choice modeling, where a random utility model is estimated, and the effect of a label can be tested. These estimations can present the same product with different labels (in our language, compare (d, x) with (d', x)). Our approach can be viewed as seeking to provide axiomatic foundations for these works, with a focus on cases in which one cannot credibly attach different labels (such as "vegetarian"/"non-vegetarian") to the same good. While the above works focus only on the consumer side, Bartling, Weber, and Yao (2015) and Pigors and Rockenbach (2016) study market behavior in laboratory experiments: they focus on, respectively, the role of consumers and firm competition in promoting socially responsible consumption.

Taking a broader perspective, the notion that consumption has socio-psychological effects has long been recognized (Veblen, 1899; Duesenberry, 1949). Frank (1985a, 1985b) highlights the role of social status, and, more recently, Heffetz (2011) studies the effects of conspicuous consumption empirically. Interdependent preferences are also at the core of Fehr and Schmidt's (1999) inequity aversion, Karni and Safra's (2002) sense of justice, as well as Ben-Porath and Gilboa's (1994) axiomatization of the Gini Index, and Maccheroni, Marinacci, and Rustichini's (2012, 2014) works on envy and pride. Conspicuous consumption can be viewed as dealing with meaning, reflecting on one's social standing and identity. Inequity aversion can similarly be conceived of as an attitude towards the value of equality. But meaning and values that are not related to social ranking are typically neglected in formal, general-purpose models of utility.

The more applied economic literature has addressed specific values more directly. For example, Barbier (1993) and Morrison (2002) study use and non-use values of wetlands. Barnes, Schier, and van Rooy (1997) examine the value of wildlife preservation, while Bedate, Herrero, and Sanz (2004) – of cultural heritage. Hornsten and Fredman (2000) and Chen and Qi (2018) deal with the value attached to forests in or near urban areas. Most of this literature relies on the controversial Contingent Valuation Method (CVM), which is based on self-reported willingness to pay.[7] Bedate, Herrero, and Sanz (2004) adopt an alternative approach suggested by Hotelling (1947), to use travel time as a way to measure the value of cultural heritage. This is indeed a measure that relies on economic choices rather than on (often hypothetical) self-report, but it clearly cannot apply to many values in question.

Medin, Schwartz, Blok, and Birnbaum (1999) argue against formal models because of the lack of attention to meaning and signification. According to their approach, decision theory lacks the *semantics* of decisions. In various questionnaires they show rather intuitive results about meaning of actions. For example, many participants in their experiments reported that they would not sell their wedding ring for any material payoff, but they would do so to save their child. Saving the life of a child would endow the sale with meaning that no material consumption can generate.

A large body of research in marketing asks what goods mean to consumers, what values they signify, and what identity they convey (Sheth, Newman, and Gross, 1991; Arnould and Thompson, 2005; He, Li, and Harris, 2012; Bajde, 2014). However, the analysis usually does not involve formal modeling in a way that can be incorporated into microeconomic theory. Calabresi (1985, 2014) discusses this point in the context of law and economics, and the degree to which economic models can capture the values society cares about.

Recent developments in behavioral economics suggest formal modeling of some related phenomena, such as the axiomatic models of Dillenberger and Sadowski (2012) on shame over selfish behavior and Evren and Minardi (2015) on warm-glow. These works are similar to ours in introducing ethical considerations into the utility function. They differ in terms of the set-up and assumptions (using menu choices and continuous preferences).

Meaning is also related to narratives one can construct. Eliaz and Spiegler (2020) deal with narratives of causality, and Glazer and Rubinstein (2021) – with stories that are sequences of events. However, both deal with narratives

[7] Throsby (2003) criticizes this measure and Kahneman and Knetch (1992) report psychological findings that undermine it.

as constructions of beliefs, whereas our focus is on their role as determinants of utility. There have been studies that challenge this dichotomy: Brunnermeier and Parker (2005) and Bracha and Brown (2012) model agents who choose not only what to do, but also what to believe. The agents we aim to model, by contrast, accept information as given. For example, a vegetarian accepts information about the ingredients of food products, and we wish to study how this information changes her consumption behavior via the value of vegetarianism, but without more involved processes such as constructing narratives or choosing what to believe.

4 Discussion

4.1 Intrinsic vs. Instrumental Values

Economic decision makers care about many non-material values. Some are vegetarian or vegan, other only consume food that is Kosher or Hallal. Many consumers would prefer to avoid any goods whose production involved child labor or abusive trade. Many are concerned about the impact that their consumption has on the environment, such as their carbon dioxide footprint, etc. We find it useful to classify the variety of values into "intrinsic" vs. "instrumental" (see Weber, 1922): a value is intrinsic if it is an end on its own, that is, if it is the ultimate carrier of meaning. Such is typically the case with vegetarianism, religious restrictions, or with avoiding child labor. By contrast, a value is instrumental if it obtains meaning only through the mediation of another value. The emission of carbon dioxide, in and of itself, will probably not be a moral issue for most consumers. But its effect on the planet, on wildlife, and on our children's future makes it meaningful.

We find that, in bold strokes, the distinction between instrumental and intrinsic values corresponds to the question of continuity at zero quantity. If a value is intrinsic, the degree to which it has been violated is of secondary importance, resulting in discontinuity at zero. By contrast, for an instrumental value the quantities of good consumed become meaningful only through some mediating mechanism, and these tend to be continuous. For example, suppose that Mary wishes to save wildlife by refraining from using plastic bags. She would be willing to pay more money in order to eat only vegetarian food and to use only biodegradable bags. But her reaction to plastic bags might still be continuous: after all, the bag itself is not a carrier of meaning. It makes a difference through a process in which the bag is disposed of, perhaps ends up in the ocean and hurts wildlife there. In other words, avoiding plastic bags is an instrumental value; correspondingly, using a single bag over the span of a year may well be identical to refraining from bags at all.

4.2 Other Distinctions

Our discussion only distinguishes between intrinsic and instrumental values, and attempts to map this conceptual distinction to the question of (dis)continuity at zero. There are, however, several other distinctions that may be conceptually insightful. First, let us revisit the production/consumption distinction. If a consumer happens to obtain a good as a gift, she might have lesser guilt feelings having to do with its production, as she has not chosen to buy it, and has not spent money to support the industry. By contrast, guilt feelings that have to do with its consumption are unaffected by the origin of the good.

Another distinction has to do with the degree to which values are translated to utilitarian calculations, and the degree that generalizations are required for the exercise. For example, a consumer might feel that she is too small to affect the number of flights, and thus, with a negligible marginal contribution, she need not feel bad about boarding a plane that is "anyway" taking off. To explain the sense of moral responsibility of such a consumer we might need to resort to a Categorical Imperative type of argument, as in the case of voting. By contrast, when a person fishes a fish for lunch, she can more directly see the consequence of her actions.

While such distinctions might, at least under some circumstances, be observable, we do not study them in this paper. Thus, we equate purchase with consumption, and do not delve into the psychological origins of the negative feelings caused by compromising values.

4.3 Unawareness

Our model assumes that the vector d is known, and, in particular, that the agent is fully aware of it. Our agents can therefore be fully rational, (provided that we do not rule out morality and values as "irrational"). We therefore assume that the values of d_i are reported whether they are positive or zero, so that there is no question of awareness of the principle, nor of uncertainty about d_i. One may extend the model to allow for the possibility that d_i isn't reported at all. This can capture a wider range of phenomena. For example, an agent who is about to take a flight might not be thinking about its environmental effects. Once airlines start reporting the environmental damage per flight (d_i) – the agent may suddenly be aware of the value-effect of her consumption decisions, and perhaps change them.

4.4 Meaning and Well-Being

The literature recognizes that well-being has both hedonic and eudaimonic determinants. The former refers to the instantaneous positive and negative sensations, whereas the latter – to a sense of meaning, self-fulfilment, and so forth. (For a review see Ryan, 2001.) Our model can be viewed as dealing with these factors as well. Consider a person who wishes to give his children broad cultural education, and, to reach this goal, is willing to give up hedonic well-being, commute longer time to work, etc. We could view this person as deriving well-being from the meaning of his material sacrifice. We could also think of him as having a *value* of enriching his children's education. Indeed, while "values" have moral connotations, in some cases it may be hard to judge whether certain cognitions are values or otherwise imbue life with meaning.

4.5 Donations

Extended versions of our model can also be used to describe the choice of donations. Assume that a consumer maximizes

$$U(x) = u(x) + v(d, x)$$

and a donation is a good x_i that does not affect the function u but that enters the function v in a way that increases well-being; that is, $u(x)$ is independent of x_i but $v(d,x)$ is increasing in it. The price for monetary donations would naturally be $p_i = 1$, and the information state d should describe what causes are served by the donated amount. In this way, the "warm glow" of donations is introduced into the utility function, but, as opposed to Evren and Minardi (2015), in this model the extent of its effect on well-being is not determined by the available menu of choices.

What form would the function $v(d,x)$ take in the case of donations? We would surely expect it to be strictly increasing in the donated amount x_i. It is less obvious whether it should be continuous at zero. On the one hand, a rational consumer should realize that donating for a cause is basically supporting an instrumental value. The act of donation itself is only a transfer of a sum of money between bank accounts, and it is hard to ascribe profound meaning to this act per se. Rather, it is the ultimate goal that this money will help support that is a carrier of meaning. The mechanism by which one's money is translated to, say, feeding hungry children, introduces continuity. On the other hand, some feeling of warm glow might result from very small amounts as well. Indeed, fund raisers might ask for a contribution, "no matter how small". And any positive donation allows one to truthfully say – to others as well as to oneself – that one has donated money. Finally, faith and religious sentiments might endow a donation with positive meaning in a way that is, to a large extent, detached from the amount donated. We thus see room both for continuous and discontinuous models of donations.

5 Appendix: Proofs of Representation Results

It will be convenient to introduce the following definition of a binary relation P on X^0:

Definition 2 *For $x, y \in X^0$, we say that xPy if there exists $z \in X^0$ and a sequence $z_n \to z$ with $(z_n) \subset X^1$ such that $x \succsim z$ and $z_n \succsim y$.*

Observe that, if we had no discontinuity between X^0 and X^1, the relation P could be expected to be equal to $\succsim$: if xPy, the conditions $z_n \to z$ and $z_n \succsim y$ would simply imply that $z \succsim y$, and $x \succsim y$ would follow by transitivity. Conversely, if $x \succsim y$, one could expect an open neighborhood of x to contain points z_n such that $z_n \succsim y$ even though $z_n \in X^1$ (for example, monotonicity would insure that this is the case). However, in the presence of discontinuity between X^1 and X^0, this is no longer the case. As explained above in the context of A4, we should expect z to be strictly better than y; indeed, intuitively, "z should be better than y at least by the cost of the principle". And the same should hold for any $x \in X^0$ such that $x \succsim z$.

Using this definition, the Archimedean axiom can be written as follows.

A4 Archimedeanity (in P terms): Let $(x_n) \subset X^0$ be such that $x_{n+1}Px_n$ (x_nPx_{n+1}) for all $n \geq 1$. Then there does not exist $\hat{x} \in X$ such that $\hat{x} \succsim x_n$ ($x_n \succsim \hat{x}$) for all $n \geq 1$.

This new formulation of A4 is simply a re-statement of the axiom in terms of the relation P. We therefore do not re-name the axiom.

5.1 Proof of Theorem 1

The proof of necessity of the axioms is straightforward and therefore omitted. To prove sufficiency, recall that $\succsim$ is continuous on X^1, and thus there exists a continuous bounded function v that represents $\succsim$ on X^1. By Theorem 2 (Online Appendix) we extend v continuously to all of X so that it represents $\succsim$ on X^0 as well. We will construct a continuous function U on X^0 that represents $\succsim$ and that also represents P by γ differences. We start out with any continuous function that represents $\succsim$ on those $x \in X^0$ for which there are no $y \in X^0$ such that xPy, use the function $v(\cdot) + \gamma$ on that set, and extend it to the rest of X^0 while respecting the representation of P by γ differences. Any element of X^1 that has a $\succsim$-equivalent in X^0 will have to have the same U value, and we will show that the resulting function is continuous on X^1 as well. Moreover, we will show that the function so constructed has a constant "jump" of γ between any sequence in X^1 that converges to a limit in X^0. We then extend it to elements of X^1 which are strictly better or strictly worse than all elements of X^0.

Proof.

Lemma 1 For $x, y \in X^0$, if xPy then $x \succ y$.

Proof: Assume not. Then there is a sequence $z_n \to z$, $(z_n) \subset X^1$, $x \succsim z$, and $z_n \succsim y$ but $y \succsim x$. By transitivity of $\succsim$, we also get $z_n \succsim x$ and by definition of P (with the same sequence $z_n \to z$), we have xPx. Define $x_n = x \in X^0$ such that $x_{n+1} P x_n$ for all n and the sequence is bounded (by $\hat{x} \equiv x$), in violation of A4.

Lemma 2 For $x, y, w \in X^0$, if xPy then (i) $y \succsim w$ implies xPw, and (ii) $w \succsim x$ implies wPy.

Proof: Suppose that $x, y, z \in X^0$ and $(z_n) \subset X^1$ are given, such that $z_n \to z$, $x \succsim z$ and $z_n \succsim y$. In case (i), $z_n \succsim y \succsim w$ and by transitivity $z_n \succsim w$, thus xPw by definition of P. As for (ii), $w \succsim x$ and $x \succsim z$ imply $w \succsim z$ and the definition of P yields wPy.

Lemma 3 For $x, y \in X^0$, if xPy, then there exists $z \in X^0$ and a sequence (z_n) with $z_n \in X^1$ such that $z_n \to z$, $x \succsim z$ and $z_n \sim y$.

Proof: Assume that $x, y \in X^0$ satisfy xPy, and that $z \in X^0$ and (z_n) with $z_n \in X^1$ satisfy $z_n \to z$, $x \succsim z$ and $z_n \succsim y$. We argue that, for each n, there exists $\alpha_n \in (0,1]$ such that $w_n \equiv \alpha_n z_n + (1 - \alpha_n) y \in X^1$ satisfies $w_n \sim y$. Indeed, if $z_n \sim y$ set $\alpha_n = 1$. Assume, then, $z_n \succ y$. If there exists $\beta \in (0,1)$ such that $y \succ \beta z_n + (1 - \beta) y$ then $z_n \succ y \succ \beta z_n + (1 - \beta) y$, with $z_n, \beta z_n + (1 - \beta) y \in X^1$, and Lemma 7 yields the existence of a point on the interval $[\beta z_n + (1 - \beta) y, z_n]$ that is indifferent to y; that point is in $[y, z_n]$ and we are done. If such a β does not exist, $\beta z_n + (1 - \beta) y \succ y$ for all $\beta > 0$. Taking a subsequence $\beta_k \searrow 0$, with $\beta_k z_n + (1 - \beta_k) y \to y$, we obtain yPy, in contradiction to Lemma 1.

Hence there are $\alpha_n \in (0,1]$ such that $w_n \equiv \alpha_n z_n + (1 - \alpha_n) y \sim y$; observe that $w_n \in X^1$ because $\alpha_n > 0$. Choose a convergent subsequence of α_n, $\alpha_{n_k} \to \alpha^*$. Then $w_{n_k} \to w^* \equiv \alpha^* z + (1 - \alpha^*) y \in X^0$. To show that $x \succsim w^*$, observe that $z_{n_k} \succsim w_{n_k}$ (because $z_{n_k} \succsim y$ and $w_{n_k} \sim y$), $z_{n_k} \to z, w_{n_k} \to w^*$, while $(z_{n_k})_k, (w_{n_k})_k \subset X^1$ and $z, w^* \in X^0$. Hence $(z_{n_k})_k \to z$ and $(w_{n_k})_k \to w^*$ are comparable and A2 yields $z \succsim w^*$ and $x \succsim w^*$ follows by transitivity.

For the rest of the appendix, we use the notation X^0_P to refer to the set defined as

$$X^0_P = \left\{ y \in X^0 \,\middle|\, \exists x \in X^0,\, xPy \right\}.$$

The strategy of the proof is to choose the continuous representation of $\succsim$ on X^0 derived from Theorem 2, take a monotone and continuous transformation thereof to obtain another representation, u, that satisfies

$$xPy \qquad \Leftrightarrow \qquad u(x) - u(y) \geq \gamma > 0$$

and then extend the function u to X^1. To this end, it will be useful to know some facts about continuous representations of $\succsim$ on X^0.

Lemma 4 Let there be given a continuous function $u : X^0 \to \mathbb{R}$ that represents $\succsim$ (on X^0). Let $y \in X^0_P$. Then there exists $\gamma(y) > 0$ such that, for every $x \in X^0$, xPy iff $u(x) - u(y) \geq \gamma(y)$. Furthermore, $\gamma(y)$ can be extended to all of X^0 so that $w \succsim y$ iff $u(w) + \gamma(w) \geq u(y) + \gamma(y)$ (for all $y, w \in X^0$).

Proof: Define $P_{y+} = \{ x \in X^0 \,|\, xPy \}$.

Case (a): Let us first consider $y \in X^0_P$ so that $P_{y+} \neq \varnothing$. Consider $u(P_{y+}) = \{ u(x) \in u(X^0) \,|\, xPy \}$. By Lemma 1, $u(y) < a$ for all $a \in u(P_{y+})$. By Lemma 2, $u(P_{y+})$ is an interval. We wish to show that it contains its infimum. Let $a = \inf u(P_{y+})$. For $k \geq 1$, let $x^k \in X^0$ be such that $a \leq u(x^k) < a + \frac{1}{k}$. Because $x^k P y$, by Lemma 3, there exist (i) $z^k \in X^0$ and (ii) $(z^k_n)_{n\geq 1}$ with $z^k_n \in X^1$ such that $z^k_n \to z^k$, $x^k \succsim z^k$ and $z^k_n \sim y$. Hence, $\forall k, l, m, n$, $z^k_n \sim z^l_m (\sim y)$. Because $(z^k_n), (z^l_m) \subset X^1$ converge to $z^k, z^l \in X^0$ respectively, A2 implies $z^k \sim z^l$. This means that $u(z^k) = u(z^l)$ $\forall k, l$ and thus $u(z^k) = a$. Hence, $a = \min u(P_{y+})$ and $a > u(y)$. It remains to define $\gamma(y) = a - u(y) > 0$. For every y such that $P_{y+} \neq \varnothing$, $\gamma(y)$ is bounded from above (by $u(x) - u(y)$ for any $x \in P_{y+}$). Observe that $\gamma(y)$ is uniquely defined $\forall y \in X^0_P$. We now show that $u + \gamma$ also represents $\succsim$ for alternatives y, w in this range.

In the construction above, $u(y) + \gamma(y) = \min u(P_{y+})$. If $w \succsim y$, Lemma 2 implies that $P_{w+} \subset P_{y+}$ and thus $\min u(P_{w+}) \geq \min u(P_{y+})$, so that $u(w) + \gamma(w) \geq u(y) + \gamma(y)$ follows. To see that the inequality is strict if $w \succ y$, let $x \in X^0$ be a $\succsim$-minimal element in P_{w+}, that is, $u(x) = u(w) + \gamma(w)$. We claim that there exists x' with $u(x') < u(x)$ such that $x'Py$ still holds (while $x'Pw$ doesn't). Because xPw, by Lemma 3 there exists $z \in X^0$ and a sequence $(z_n)_{n\geq 1}$ with $z_n \in X^1$ such that $z_n \to z$, $x \sim z$ and $z_n \sim w$ (and $x \sim z$ follows from the minimality of x). Hence, $z_n \succ y$. As in the proof of Lemma 3, for each z_n we can find $\alpha_n \in (0,1]$ such that $t_n \equiv \alpha_n z_n + (1-\alpha_n) y \in X^1$ satisfies $t_n \sim y$ (or else yPy would follow).

Taking a convergent subsequence of α_n, say $\alpha_{n_k} \to \alpha^*$, we have $t_{n_k} \to t^* \equiv \alpha^* z + (1-\alpha^*) y \in X^0$. We thus have two sequences $(z_{n_k}), (t_{n_k}) \subset X^1$, with $z_{n_k} \sim w \succ y \sim t_{n_k}$ and $z_{n_k} \to z, t_{n_k} \to t^*$ with $z, t^* \in X^0$. Observe that $(z_{n_k}) \to z, (t_{n_k}) \to t^*$ are comparable. Hence A3 implies that $z \succ t^*$. Thus $\exists x' \in X^0$ with $u(x') \in (u(t^*), u(z))$. As z (and x) was selected to have the lowest possible u in $u(P_{w+})$, $x'Pw$ doesn't hold, while $x'Py$ does.

Case (b): For $y \in X^0 \backslash X_P^0$ we set $\gamma(y)$ to be a constant, defined as follows. Let $\bar{u} = \sup_{z\in X_P^0} u(z)$. This sup may or may not be a max.[8] Define $\gamma(y) = \lim_{n\to\infty} \sup \left\{ \gamma(z) \,\middle|\, \bar{u} - \frac{1}{n} < u(z) \leq \bar{u} \right\}$. It is finite because $\gamma(y)$ is bounded from above by $(u(x) - \bar{u} + 1)$ $\forall x \in P_{z+}$. Because $\gamma(y)$ is constant for all $y \in X^0 \backslash X_P^0$, and u represents $\succsim$ for alternatives y, w in this range, so does $u + \gamma$. Next, observe that $\sup_{z\in X_P^0} [u(z) + \gamma(z)] = \sup_{z\in X^0} u(z)$ and, for $y \in X^0 \backslash X_P^0$ and $w \in X_P^0$ we have $u(y) + \gamma(y) \geq \sup_{z\in X^0} u(z) \geq u(w) + \gamma(w)$ and $u(y) + \gamma(y) > u(w) + \gamma(w)$. That is, the value $\sup_{z\in X^0} u(z)$ might be obtained by $u(\cdot) + \gamma(\cdot)$ on X^0 or on $X^0 \backslash X_P^0$ but not on both, so that $u + \gamma$ represents $\succsim$ on the entire range.

Lemma 5 Let there be given a continuous function $u : X^0 \to \mathbb{R}$ that represents $\succsim$ (on X^0). There exists a continuous function $\phi : u(X^0) \to \mathbb{R}$ such that, for every $x \in X^0, y \in X_P^0$, xPy iff $u(x) - u(y) \geq \phi(u(y))$ and $u(\cdot) + \phi(u(\cdot))$ also represents $\succsim$ on X^0.

Proof: Use Lemma 4 to define $\gamma : X^0 \to \mathbb{R}$ such that $u(\cdot) + \gamma(\cdot)$ represents

[8]For example, for $n = 2$, $X = [0,10]^2$ and $d = (1,0)$ consider $u_1(x_1, x_2) = x_2 + x_1$ and $u_2(x_1, x_2) = x_2 + (x_1 - 1)^2$. In both cases define the relation by the function u_i and $\gamma = 1$. In the case of u_1 the relation P is a closed subset of $X^0 \times X^0$ an $\bar{u} = 9$ is the max of $u(z)$ over X_P^0, whereas for u_2 P isn't closed, and the point $(9,0)$ is not in X_P^0, leaving $\bar{u} = 9$ the sup of the utility in X_P^0.

$\succsim$ on X^0 and xPy iff $u(x) - u(y) \geq \gamma(y)$ whenever $y \in X_P^0$ as above. Observe that, $\forall y, w \in X^0$, we have $w \succsim y$ iff $u(w) + \gamma(w) \geq u(y) + \gamma(y)$. Hence $w \sim y$ implies $u(w) + \gamma(w) = u(y) + \gamma(y)$ and, since $u(w) = u(y)$ also holds in this case, $\gamma(w) = \gamma(y)$. It follows $\exists \phi : \mathbb{R} \to \mathbb{R}$ such that $\gamma(y) = \phi(u(y))$, uniquely defined for all values $\bar{u} = u(y)$ such that $y \in X_P^0$. To show that ϕ is continuous on that range, let there be given $\bar{u} \in range(u)$ and $(u^k)_{k\geq 1}$ so that $u^k \in range(u)$ and $u^k \to \bar{u}$ (as $k \to \infty$). If $\phi(u^k) \to \phi(\bar{u})$ fails to hold, there exists $\varepsilon > 0$ such that (i) there are infinitely many k's for which $\phi(u^k) < \phi(\bar{u}) - \varepsilon$ or (ii) there are infinitely many k's for which $\phi(u^k) > \phi(\bar{u}) + \varepsilon$.

In case (i), let $y \in u^{-1}(\bar{u})$ and $y^k \in u^{-1}(u^k)$ for k from some k_0 on (obviously, with $y \in X_P^0$ and $y^k \in X_P^0$ for all k). As u is continuous, we can also choose such a y and a corresponding sequence so that $y^k \to y$. Let $t, t' \in X^0$ be such that $u(y) + \phi(u(y)) = u(t) > u(t') > u(y^k) + \phi(u(y^k))$ for all $k \geq k_0$, so that $t \succ t'$, tPy^k,$t'Py^k$ for all k, tPy but not $t'Py$. As tPy we can select a sequence $(z_n) \subset X^1$ with $z_n \to z \in X^0$, $t \succsim z$ and $z_n \sim y$. By the choice of t (as a u-minimal element such that tPy), $u(t) = u(z)$. As $t'Py^k$, there is, $\forall k$, a sequence $(w_n^k) \subset X^1$ such that $w_n^k \to w^k \in X^0$, $t' \succsim w^k$ and $w_n^k \sim y^k$. As above, select a convergent subsequence of the diagonal to get a sequence $(w_n^n) \subset X^1$ such that $w_n^n \to w \in X^0$, $t' \succsim w \in X^0$ and $w_n^n \sim y^n$. By transitivity, $z \sim t \succ t' \succsim w$. Observe that $z_n \to z$ and $w_n^n \to w$ are comparable, and we also have $z \succ w$. Use Lemma 10 (Online Appendix) for $y_n = y^n \to y$ and $x_n = x = y$. Because $x_n, y_n, x, y \in X^0$, $y_n \to y$ and $x_n \to y$ are also comparable. Lemma 10 implies $y \succ y$, a contradiction.

In case (ii) select $t, t' \in X^0$ be such that $u(y) + \phi(u(y)) = u(t) < u(t') < u(y^k) + \phi(u(y^k))$ $\forall k \geq k_0$, so that $t' \succ t$, tPy and $t'Py$ hold, but tPy^k,$t'Py^k$ do not hold for any k. For each k, $\exists t^k$ such that $u(t^k) = u(y^k) + \phi(u(y^k))$ (a u-minimal element satisfying t^kPy^k). Let $(z_n^k) \subset X^1$ be such that $z_n^k \to z^k \in X^0$, $t^k \succsim z^k$ and $z_n^k \sim y^k$. Let $(z_n) \subset X^1$ be such that $z_n \to z \in X^0$, $t \succsim z$ and $z_n \sim y$. By the choice of t,(t^k) as minimal elements, $t \sim z$ and $t^k \sim z^k$. Select a convergent subsequence of $z_k^k \to z^* \in X^0$. Because $z^k \succsim t'$ (and $z^k \in X^0$) we have $z^* \succsim t' \succ t$. Again, contradiction follows from Lemma 10.

To complete the proof, use Theorem 2 (Online Appendix) to have a continuous and bounded function $v : X \to \mathbb{R}$ that represents $\succsim$ on X^0 and on X^1. By A5, it isn't constant on X^0. Assume, w.l.o.g., that $\inf_{x\in X^0} v(x) = 0$ and $\sup_{x\in X^0} v(x) = 1$. Let $b = \inf_{x\in X} v(x)$ and $a = \sup_{x\in X} v(x)$ so that $b \le 0 \le 1 \le a$. Next, define a continuous $u : X \to \mathbb{R}$ and $\gamma > 0$ such that $U(x) = u(x) - \gamma \mathbf{1}_{\{x\in X^1\}}$ represents $\succsim$. We first define $U = u$ on X^0, and $\Delta > 0$ such that u represents $\succsim$ on X^0, and (u, Δ) jointly represent P on X^0 by $[xPy \Leftrightarrow u(x) - u(y) \ge \Delta]$, and then define u on X^1 and γ.

Step 1: Definition of $U = u$ on X^0

If $P = \varnothing$ define $u = v$ and $\Delta = 2$. Clearly, the representation of P holds. Otherwise, if $P \ne \varnothing$, we construct a partition of X^0 into countably many subsets X_k^0 for $k \in \mathbb{Z}$ such that, if $x \in X_k^0$ and $y \in X_l^0$, then $k > l + 1$ implies xPy and $k \le l$ implies $\neg(xPy)$. First, we define a function $S : X^0 \times X^0 \to \mathbb{Z}$ to be the maximal k such that there are $z_0 = x, z_k = y, z_i P z_{i+1}$ for $\forall i \le k - 1$. For $x \succsim y \succsim z$, we have $S(z, y) + S(y, x) \le S(z, x) \le S(z, y) + S(y, x) + 1$. For $x, y \in X^0$ with $y \succ x$, set $S(y, x) = -S(x, y) - 1$ so that $S(y, x) + S(x, y) = -1$ for all $x \nsim y$. We finally define u on X^0. Distinguish between two cases:

Case 1a: $\forall x \in X^0\ \exists y \in X^0$ such that xPy. In this case, should the representation of P by Δ hold, u should be unbounded from below. Select an $x_0 \in X^0$ with $P_{x_0+} \ne \varnothing$. For $k \in \mathbb{Z}$, let $X_k^0 = \{ y \in X^0 \mid S(x_0, y) = k \}$. For $y \in X_0^0$ (that is, $y \succsim x_0$ but not yPx_0), set $u(y) = v(y) - v(x_0)$ (in particular, $u(x_0) = 0$). Let $\Delta = \sup_{X_0^0} u(y)$. By Lemma 4, $\Delta > 0$. Once u is defined for all $y \in X_k^0$ for $k \ge 0$, extend it to X_{k+1}^0 as follows: $\forall x \in X_{k+1}^0\ \exists y \in X_k^0$ such that $v(x) = v(y) + \phi(v(y))$ where ϕ is the function constructed in Lemma 5 for v (and by Lemma 5, this is the highest y that satisfies xPy). Set $u(x) = u(y) + \Delta$. Similarly, if u is defined for all $y \in X_k^0$ for $k \le 0$, extend it to X_{k-1}^0 by $u(x) = u(y) - \Delta$ for $x \in X_{k-1}^0$ and $y \in X_k^0$ such that $v(y) = v(x) + \phi(v(x))$. It is straightforward to verify that u so constructed is a continuous strictly monotone transformation of v and thus represents $\succsim$ on X^0. Define also $U = u$ on X^0.

Case 1b: $\exists x \in X^0$ such that, $\forall y \in X^0$ we have $\neg(xPy)$. If there exists a v-(equivalently, a $\succsim$-) minimal element in X^0, denote it by x_0 and proceed as in Case 1a. If not, let $\alpha = \sup \{ v(x) \mid x \in X^0, \nexists y \in X^0, \quad xPy \}$ so that $v(x) > \alpha$

implies ($\exists y \in X^0$, xPy) and $v(x) < \alpha$ implies ($\forall y \in X^0$, $\neg(xPy)$). If $\alpha = 0$, in the absence of a minimal element, then we are in Case 1a (where each $x \in X^0$ P-dominates at least one other element). Hence $\alpha > 0$. Define $u(x) = v(x)$ for all x with $v(x) \leq \alpha$ and $\Delta = \alpha$. For x with $v(x) > \alpha$ we repeat the construction above, with X^0_k including all elements $x \in X^0$ for which the maximal decreasing P-chain is of length k.

Step 2: Definition of u on X^1 and of γ

To extend the function to all of X, partition X^1 into three sets, $X^{1\sim}$ – the elements that have a $\sim$-equivalent in X^0, and $X^{1\prec}$ ($X^{1\succ}$) – those that are worse (better) than all elements in X^0. If $X^{1\sim} \neq \varnothing$, each of $X^{1\prec}$,$X^{1\succ}$ may be empty or not. However, if $X^{1\sim} = \varnothing$ we have to have $X^{1\prec} \neq \varnothing$: otherwise ($X^{1\succ} = X^1$) all $x \in X^1$ and $y \in X^0$ will satisfy $x \succ y$ and yPy would follow. Further, in this case, since $X^{1\sim} = \varnothing$ and $X^{1\prec} \neq \varnothing$ we also have $X^{1\succ} = \varnothing$, by Lemma 7. We will therefore split the definition according to the emptiness of $X^{1\sim}$.

Case 2a: $X^{1\sim} = \varnothing$. In this case we have $X^{1\sim} = X^{1\succ} = \varnothing$ as well as $P = \varnothing$ (as no element in X^1 is ranked as high as any in X^0). Define $u(x) = v(x)$ $\forall x \in X^1 = X^{1\prec}$, and set $\gamma = 2(a - b) \geq \Delta$. On X^1, $U(x) = v(x) - \gamma$. Thus $u = v$ is a continuous function on all of X, U represents $\succsim$ on X^0 as well as on X^1, and it also satisfies $U(x) < U(y)$ for every $x \in X^1$ and every $y \in X^0$.

Case 2b: $X^{1\sim} \neq \varnothing$. We first define U that would represent $\succsim$ on the entire space, and then find the $\gamma > 0$ such that $u(x) = U(x) + \gamma \mathbf{1}_{\{x \in X^1\}}$ is continuous. For $x \in X^{1\sim}$, let $y \in X^0$ be such that $x \sim y$ and define $U(x) = U(y)$. This function represents $\succsim$ on $X^0 \cup X^{1\sim}$. We wish to show that it is continuous on $X^{1\sim}$.

Claim: $U : X^0 \cup X^{1\sim} \to \mathbb{R}$ is continuous (also) on $X^{1\sim}$.

Proof: Let there be $(x_n) \to x$ in $X^{1\sim}$ and select corresponding $(y_n), y$ in X^0 (so that $x \sim y$ and $x_n \sim y_n$). Assume first that $x_1 \succ x$ and that $x_1 \succsim x_n \succsim x$ $\forall n$. We claim that $u(x_n) \to u(x)$. A symmetric argument would apply to the case $x_1 \prec x$ and ($x_1 \precsim x_n \precsim x$ $\forall n$) and the combination of the two would complete the proof. We thus have $x_1 \sim y_1 \succsim x_n \sim y_n \succsim x \sim y$ $\forall n$. By Lemma 7 $\exists \alpha_n \in [0, 1]$ such that $\hat{y}_n \equiv \alpha_n y_1 + (1 - \alpha_n) y \sim y_n$. By convexity of X^0, $\hat{y}_n \in X^0$. Thus, $U(x_n) = U(y_n) = U(\hat{y}_n)$ and $U(x) = U(y)$. Select a convergent subsequence

$(n_k)_k$ such that $\hat{y}_{n_k} \to y^* \in X^0$. As U is continuous on X^0, we have $U(\hat{y}_{n_k}) \to U(y^*)$. Because $x_{n_k} \to x$ are in $X^{1\sim}$ and $\hat{y}_{n_k} \to y^*$ are in X^0, the two sequences are comparable and A2 implies that $x \sim y^*$ and thus also $y \sim y^*$. It follows that $U(x) = U(y) = U(y^*) = \lim U(\hat{y}_{n_k}) = \lim U(x_{n_k})$. □

Let $v_* = \inf_{x\in X^{1\sim}} v(x)$ and $v^* = \sup_{x\in X^{1\sim}} v(x)$. Recall that v is bounded (by b, a) and thus $b \le v_* \le v^* \le a$. Denote $u^* = \sup_{x\in X^{1\sim}} U(x)$ and $u_* = \inf_{x\in X^{1\sim}} U(x)$ (which can be ∞, $-\infty$, respectively).

On $X^{1\sim}$, both v and U represent $\succsim$, and are continuous. Thus there exists a continuous, strictly increasing $\psi : (v_*, v^*) \to (u_*, u^*)$ such that, $\forall x \in X^{1\sim}$, $U(x) = \psi(v(x))$ and $\lim_{v\searrow v_*} \psi(v) = u_* \quad \lim_{v\nearrow v^*} \psi(v) = u^*$. Further, if v_* is obtained by v on $X^{1\sim}$, $u_* > -\infty$ and we can define $\psi(v_*) = u_*$, and, similarly, $\psi(v^*) = u^*$ in case $v^* = \max_{x\in X^{1\sim}} v(x)$ (and $u^* < \infty$).

Next, extend ψ to the entire range of v on X^1. Consider first $v > v^*$. If $X^{1\succ} = \varnothing$, then v on X^1 is bounded above by v^*, and the extension of ψ to this range is immaterial. Otherwise, that is, $X^{1\succ} \neq \varnothing$, A4 implies $U(x) < \infty$ $\forall x \in X^0$ and hence $u^* < \infty$. Set $\psi(v) = (v - v^*) + u^*$ $\forall v > v^*$, representing $\succsim$ on $X^{1\succ}$. Similarly, consider $v < v_*$. If $X^{1\prec} = \varnothing$, then v on X^1 is bounded below by v_*, and the extension of ψ to this range is immaterial. Otherwise, that is, $X^{1\prec} \neq \varnothing$, we know, by A4, that $U(x) > -\infty$ for all $x \in X^0$ and this means that $u^* > -\infty$. Hence we can set $\psi(v) = (v - v_*) + u_*$ for all $v < v_*$. Thus, $U(x) = \psi(v(x))$ is well defined for all $x \in X^1$; combined with the definition of U on X^0, we know that (i) U represents $\succsim$ on the entire space X; (ii) U is continuous on each of X^0 and X^1. It remains to define $\gamma > 0$ and show that, for that γ, $u(x) = U(x) + \gamma \mathbf{1}_{\{x\in X^1\}}$ is continuous on the entire space. We set γ to be equal to Δ as defined in Step 1.

Claim: $u : X \to \mathbb{R}$ is continuous on X.

Proof: We only need to consider sequences $(x_n) \subset X^1$ that converge to $x \in X^0$. Let there be given such a sequence $(x_n) \subset X^1$ with $x_n \to x \in X^0$. Distinguish between two cases:

Case 2b(i): $\exists y \in X^0$, xPy. Assume w.l.o.g. that $u(y) = u(x) - \gamma$, i.e., that y is a u-maximal element with xPy. There exists a sequence $(x'_n) \subset X^1$ with $x'_n \to x \in X^0$ and $x'_n \sim y$ so that $U(x'_n) = U(y) = u(y)$ and, U on X^1

being a continuous transformation of v, where the latter is continuous on all of X, we also have $U(x_n) \to \lim_n U(x'_n) = u(y) = u(x) - \gamma = U(x) - \gamma$. Hence $u(x_n) = U(x_n) + \gamma \to U(x) = u(x)$ as required.

Case 2b(ii): $\nexists\, y \in X^0$ with xPy. By the definition of $u = U$ on X^0 in Step 1, we are in Case 1b and $u(x) = U(x) = v(x)$. Consider the sequence (x_n). Because it is convergent, and v is continuous on X, $\exists \lim_n v(x_n)$ $(= v(x))$. On X^1 $U(\cdot) = \psi(v(\cdot))$ is continuous, thus $\exists \lim_n U(x_n)$. The limit $v(x) = \lim_n v(x_n)$ cannot exceed v_* (if it did, $\exists y \in X^0$ such that $x_n \gtrsim y$ for infinitely many n's, and xPy would follow). However, in the domain $v \leq v_*$ we have $\psi(v) = (v - v_*) + u_* = v - (v_* - u_*)$. Further, in this case (corresponding to Case 1b in the definition of u on X^0), $u_* = \inf_{x \in X^0} v(x) = 0$ while $v_* = \inf_{x \in X^{1\sim}} v(x) = \Delta = \gamma$. It follows that

$$\lim_n U(x_n) = \lim_n \psi(v(x_n)) = \lim_n v(x_n) - \gamma = v(x) - \gamma$$

and thus $u(x_n) = U(x_n) + \gamma \to v(x) = u(x)$ and continuity is established. ■

5.2 Proof of Corollary 1

We re-state axioms A6 in terms of the relation P and show that, when X is compact, it implies A4. First note that the axiom can be written as

A6 Lipschitz: There exists $\delta > 0$ such that, for every $x, y \in X^0$, if xPy then $\|x - y\| > \delta$.

Assume that X is compact, which implies that X^0 is compact as well. We wish to show that no infinite decreasing P chain can be bounded from below, nor can an infinite increasing P chain be bounded from above. However, A6 would make a stronger claim, namely, that there are no infinite P chains (neither increasing nor decreasing). Indeed, Lemma 2 implies that P is transitive. Had there been an infinite P chain, we would have to find two elements, say x_i and x_j such that $x_i P x_j$ (with $i > j$ for a decreasing P chain and $i < j$ for an increasing one) while they are in a δ-neighborhood of each other, in contradiction to A6. □

6 References

Arnot, C., P. C. Boxall, and S. B. Cash (2006), "Do Ethical Consumers Care about Price? A Revealed Preference Analysis of Fair Trade Coffee Purchases", *Canadian Journal of Agricultural Economics*, **54**: 555-565.

Arnould, E. J., and C. J. Thompson (2005), "Consumer Culture Theory (CCT): Twenty Years of Research", *Journal of Consumer Research*, **31**: 868-882.

Auger, P., P. Burke, T. M. Devinney, and J. J. Louviere (2003), "What Will Consumers Pay for Social Product Features?", *Journal of Business Ethics* **42**: 281-304.

Bajde, D. (2014), "Consumer Culture Theory: Ideology, Mythology and Meaning in Technology Consumption", *International Journal of Actor-Network Theory and Technological Innovation*, **6**: 10-25.

Barbier, E. B. (1993), "Sustainable Use of Wetlands Valuing Tropical Wetland Benefits: Economic Methodologies and Applications", *The Geographical Journal*, **159**: 22-32.

Barnes, J. I., C. Schier, and G. van Rooy (1997), "Tourists' Willingness to Pay for Wildlife Viewing and Wildlife Conservation in Namibia", DEA Research Discussion Paper no. 15.

Bartling, B., R. A. Weber, and L. Yao (2015), "Do Markets Erode Social Responsibility?", *The Quarterly Journal of Economics*, **130**: 219-266.

Basu, A. K. and R. L. Hicks (2008), "Label Performance and the Willingness to Pay for Fair Trade Coffee: A Cross-National Perspective", *International Journal of Consumer Studies*, **32**: 470-478.

Barnett, C., P. Cloke, N. Clarke, and A. Malpass (2005), "Consuming Ethics: Articulating the Subjects and Spaces of Ethical Consumption", *Antipode*, **37**: 23-45.

Bedate, A., L. C. Herrero, and J. A. Sanz (2004), "Economic Valuation of the Cultural Heritage: Application to Four Case Studies in Spain", *Journal of*

Cultural Heritage, **5**: 101-111.

Beja, A. and I. Gilboa (1992), "Numerical Representations of Imperfectly Ordered Preferences (A Unified Geometric Exposition)", *Journal of Mathematical Psychology*, **36**: 426-449.

Ben-Porath, E., and I. Gilboa (1994), "Linear Measures, the Gini Index and the Income-Equality Tradeoff", *Journal of Economic Theory*, **64**: 443-467.

Blume, L., A. Brandenburger, and E. Dekel (1991), "Lexicographic Probabilities and Equilibrium Refinements", *Econometrica*, **59**: 81-98.

Bracha, A. and D. J. Brown (2012), "Affective Decision Making: A Theory of Optimism Bias", *Games and Economic Behavior*, **75**: 67-80.

Brunnermeier, M. K. and J. A. Parker (2005), "Optimal Expectations", *The American Economic Review*, **95**: 1092-1118.

Calabresi, G. (1985), Ideals, Beliefs, Attitudes, and the Law. Syracuse, NY: Syracuse University Press.

Calabresi, G. (2014), "Of Tastes and Values", Yale Law School Research Paper No. 500.

Chen, B. and X. Qi (2018), "Protest response and contingent valuation of an urban forest park in Fuzhou City, China", *Urban Forestry and Urban Greening*, **29**: 68-76.

Debreu, G. (1954), Representation of a preference ordering by a numerical function. In: Thrall, R., Coombs, C., Davis, R. (eds.) Decision Processes. Wiley, New York.

Debreu, G. (1959), Theory of Value. Wiley, New York.

De Pelsmacker, P., L. Driesen, and G. Rayp (2005), "Do Consumers Care about Ethics? Willingness to Pay for Fair-Trade Coffee", *Journal of Consumer Affairs*, **39**: 363-385.

Dillenberger, D. and P. Sadowski (2012), "Ashamed to be Selfish", *Theoretical Economics*, **7**: 99-124.

Duesenberry, J. S. (1949), *Income, Saving and the Theory of Consumer Behaviour.* Cambridge: Harvard University Press.

Eliaz, K. and R. Spiegler (2020), "A Model of Competing Narratives ", *American Economic Review*, **110**: 3786-3816.

Enax L., V. Krapp, A. Piehl, and B. Weber (2015), "Effects of Social Sustainability Signaling on Neural Valuation Signals and Taste-Experience of Food Products", *Frontiers in Behavioral Neuroscience,* **9**: article 247 (1-15).

Evren, O. and S. Minardi (2015), "Warm-Glow Giving and Freedom to be Selfish", *The Economic Journal*, **127**: 1381-1409.

Fehr, E. and K. M. Schmidt (1999), "A Theory of Fairness, Competition, and Cooperation", *The Quarterly Journal of Economics*, **114**: 817-868.

Frank, R. H. (1985a), *Choosing the Right Pond: Human Behavior and the Quest for Status*. Oxford: Oxford University Press.

Frank, R. H. (1985b), "The Demand for Unobservable and Other Nonpositional Goods". *American Economic Review*, **75**: 101-116.

Glazer, J. and A. Rubinstein (2021), "Story Builders", *Journal of Economic Theory*, **193**, 105211.

Hainmueller, J., M. J. Hiscox, and S. Sequeira (2015), "Consumer Demand for Fair Trade: Evidence from a Multistore Field Experiment", *The Review of Economics and Statistics*, **97**: 242-256.

He, H., Y. Li, and L. Harris (2012), "Social Identity Perspective on Brand Loyalty", *Journal of Business Research*, **65**: 648-657.

Heffetz, O. (2011), "A Test of Conspicuous Consumption: Visibility and Income Elasticities", *The Review of Economics and Statistics*, **93**: 1101-1117.

Hornsten, L. and P. Fredman (2000), "On the Distance to Recreational Forests in Sweden", *Landscape and Urban Planning*, **51**: 1-10.

Hotelling, H. (1947), "The Economics of Public Recreation", The Prewitt Report, National Parks Service, Washington.

Loureiro, M. and J. Lotade (2005), “Do Fair Trade and Eco-Labels in Coffee Wake Up the Consumer Conscience?”, *Ecological Economics*, **53**: 129-138.

Kahneman D. and J. Knetch (1992), “Valuing Public Goods: The Purchase of Moral Satisfaction”, *Journal of Environmental Economics and Management*, **22**: 57-70.

Karni, E. and Z. Safra (2002), “Individual Sense of Justice: A Utility Representation”, *Econometrica,* **70**: 263-284.

Maccheroni, F., M. Marinacci, and A. Rustichini (2012), “Social Decision Theory: Choosing within and between Groups”, *Review of Economic Studies*, **79**: 1591-1636.

Maccheroni, F., M. Marinacci, and A. Rustichini (2014), “Pride and Diversity in Social Economies”, *American Economic Journal: Microeconomics*, **6**: 237-271.

Medin, D. L., H. C. Schwartz, S. V. Blok, and L. A. Birnbaum (1999), “The Semantic Side of Decision Making”, *Psychonomic Bulletin and Review*, **6**: 562-569.

Morrison, M. (2002), “Understanding Local Community Preferences for Wetland Quality”, *Ecological Management and Restoration*, **2**: 127-134.

Pigors M. and B. Rockenbach (2016), “Consumer Social Responsibility”, *Management Science*, **62**: 3123-3137.

Prasad, M., H. Kimeldorf, R. Meyer, and I. Robinson (2004), “Consumers of the World Unite: A Market-Based Response to Sweatshops”, *Labor Studies Journal*, **29**: 57-80.

Rubinstein, A. (1998), “Definable Preferences: An Example”, *European Economic Review*, **42**: 553-560.

Ryan, R. M. (2001), “On Happiness and Human Potentials: A Review of Research on and Eudaimonic Well-Being”, *Annual Review of Psychology*, **52**: 141-166.

Sheth, J., B. I. Newman, and B. L. Gross (1991), "Why We Buy What We Buy: A Theory of Consumption Values", *Journal of Business Research*, **22**:159-170.

Throsby, D. (2003), "Determining the Value of Cultural Goods: How Much (or How Little) Does Contingent Valuation Tell Us?", *Journal of Cultural Economics*, **27**: 275-285.

Veblen, T. (1899), *The Theory of the Leisure Class: An Economic Study in the Evolution of Institutions.* New York: Macmillan. Also: 1994 Dover paperback edition,1994 Penguin Classics edition.

Weber, M. (1922), *Economy and Society.* University of California Press (1978).

7 Online Appendix

This online appendix is organized in two subsections. Subsection 7.1 contains an auxiliary result. Subsection 7.2 presents some examples showing that all axioms presented in the paper are independent of each other.

7.1 An Auxiliary Result

This appendix presents and proves the following auxiliary result used in the proof of Theorem 1.

Theorem 2 *Let $\succsim$ on X satisfy A1-A3. Then, a bounded and continuous function $u : X^1 \to \mathbb{R}$ that represents $\succsim$ on X^1 has a unique continuous extension to (all of) X. This extension represents $\succsim$ also on X^0.*

Note that the theorem does not state that the extended u represents $\succsim$ on X in its entirety. Indeed, the continuity axioms do not state that preferences change continuously along a sequence that crosses from X^1 to X^0, and thus a utility function that is continuous on the entire space cannot be expected to represent preferences across the two subspaces.

Proof of Theorem 2. Without loss of generality we assume that d isn't identically 0 not identically 1, so that $X^0, X^1 \neq \varnothing$. Note that, due to convexity of X^1, X^0 is included in the closure of X^1.

We start with a few lemmas. Throughout we assume that $\succsim$ on X satisfies A1-A3. (Note, however, that the first three lemmas do not make use of A3).

Lemma 6 Let there be a sequence $x_n \to x$. Assume that $[(x_n) \subset X^0$ and $x \in X^0]$ or $[(x_n) \subset X^1$ and $x \in X^1]$. Then, for all $y \in X$, if $x_n \succsim y$, then $x \succsim y$ and if $y \succsim x_n$, then $y \succsim x$.

Proof: Define $y_n = y$ for all $n \geq 1$. Note that the sequences $x_n \to x$ and $y_n \to y$ are comparable (satisfying Condition A), and apply A2. □

Lemma 7 Let there be $x, y, z \in X$ with $x \succ y \succ z$. Assume that $x, z \in X^0$ or that $x, z \in X^1$. Then there exists $\alpha \in [0,1]$ such that $y \sim \alpha x + (1-\alpha) z$.

Proof: The argument is familiar, and we mention it explicitly to point out that it does not depend on monotonicity or openness conditions. Let there be $x, y, z \in X$ with $x \succ y \succ z$ and assume without loss of generality that $x, z \in X^0$ (the argument is identical for X^1). Define

$$\begin{aligned} A^- &= \{\alpha \in [0,1] \mid y \succ \alpha x + (1-\alpha) z\} \\ A^+ &= \{\alpha \in [0,1] \mid \alpha x + (1-\alpha) z \succ y\} \end{aligned}$$

and we have $A^- \cap A^+ = \emptyset$, with $1 \in A^+$ and $0 \in A^-$. Consider $\alpha^* = \inf A^+$ and define $x^* = \alpha^* x + (1-\alpha^*) z$. We wish to show that it is the desired α, so that $\alpha^* \notin A^- \cup A^+$ and $y \sim x^*$ holds. Suppose that this is not the case. If $\alpha^* \in A^-$ (and $y \succ x^*$), we can choose a sequence $\alpha_n^+ \in A^+$ with $\alpha_n^+ \searrow \alpha^*$. Then $x_n = \alpha_n^+ x + (1-\alpha_n^+) z \in A^+ \to x^*$. Importantly, X^0 is convex. Hence $x_n \in X^0$ for all n and $x^* \in X^0$ as well. Lemma 6 implies that $x^* \succsim y$, a contradiction. Similarly, if $\alpha^* \in A^+$ (and $x^* \succ y$), then $\alpha^* = \min A^+$ and we must have $\alpha^* > 0$ as $0 \in A^-$, in which case we can choose a sequence $\alpha_n^- \in A^-$ with $\alpha_n^- \nearrow \alpha^*$. Then, Lemma 6 implies that $y \succsim x^*$, again a contradiction. Hence $y \sim x^*$.

The argument holds also for X^1 since it is a convex set as well. □

We also note the following.

Lemma 8 For all comparable sequences $\xi_n \to \xi$ and $\eta_n \to \eta$, if $\xi \succ \eta$, then there exists an $N > 0$ such that

$$\xi_n \succ \eta_m \quad \forall n, m > N.$$

Proof: If the conclusion does not hold, for $N_1 = 1$ we have n_1, m_1 such that $\eta_{m_1} \succcurlyeq \xi_{n_1}$. Set $N_2 = \max(n_1, m_1)$ and find $n_2, m_2 > N_2$ such that $\eta_{m_2} \succcurlyeq \xi_{n_2}$. Continuing this way, we generate two subsequences $(n_k, m_k)_k$ such that $\eta_{m_k} \succcurlyeq \xi_{n_k}$ for all k, with $\xi_{n_k} \to \xi$ and $\eta_{m_k} \to \eta$ being comparable (as subsequences of comparable sequences with these limits). A2 would then imply $\eta \succcurlyeq \xi$, a contradiction. □

Two implications of the A3 (in the presence of A1, A2) will be useful to state explicitly.

Lemma 9 For all comparable sequences $x_n \to x$ and $y_n \to x$, and all $z, w \in X$, if ($x_n \succsim z$ and $w \succsim y_n$) then $w \succsim z$.

Proof: Let there be given comparable sequences $x_n \to x$ and $y_n \to x$ as well as $z, w \in X$ such that $x_n \succsim z$ and $w \succsim y_n$. We need to show that $w \succsim z$. Assume, to the contrary, that $z \succ w$. Define $y = x$. With $x_n \succsim z \succ w \succsim y_n$ we can apply A3 and conclude that $x \succ y$ which is impossible as $y = x$. Thus we rule out the possibility $z \succ w$ and conclude that $w \succsim z$ as required. □

The following lemma is not needed for Theorem 2 but it is used in the proof of Theorem 1. It is similar to A3 and can easily be shown to imply it. Thus the lemma shows that, in the presence of A1 and A2, the two conditions are equivalent.

Lemma 10 For all pairs of comparable sequences, ($x_n \to x$ and $y_n \to y$) and ($z_n \to z$ and $w_n \to w$), if (i) $z \succ w$; and (ii) $x_n \succsim z_n$; $w_n \succsim y_n$ for all n, then $x \succ y$.

Proof: Assume, then, that ($x_n \to x$ and $y_n \to y$) and ($z_n \to z$ and $w_n \to w$), are given, such that (i) $z \succ w$; and (ii) $x_n \succsim z_n$; $w_n \succsim y_n$ for all n. We split the argument depending on the reason that $z_n \to z$ and $w_n \to w$ are comparable. Assume, first, that they satisfy Condition A, that is, that $(z_n) \subset X^i, z \in X^i$ and $(w_n) \subset X^j, w \in X^j$ for $i, j \in \{0, 1\}$. In this case, because the limit of each sequence $(z_n), (w_n)$ belongs to the same space X^i as the sequence itself, we also have, w.l.o.g., $z_n \succ w$ and $z \succ w_n$ for all n. (Otherwise, we can apply A2 to the relevant sequence and to a constant sequence and derive $w \succsim z$ from A2.) Next, consider a specific $n > N$. If there are infinitely many indices $n_k > n$ such that $z_{n_k} \succsim z_n$, let n be the minimal index with this property, and, for that n, set $z^* = z_n$ and restrict attention to the subsequence $(n_k)_k$. Clearly, $x_{n_k} \succsim z_{n_k} \succsim z_n = z^*$. If not, then for every $n > N$ there is $l_n > 0$ such that, for all $m > n + l_n$, we have $z_n \succ z_m$. In that case we can select a subsequence (z_{n_k}) such that $z_{n_k} \succ z_{n_{k+1}}$. As $(z_{n_k}) \to z$ and belongs to the same space (as z), we can compare it to the sequence that equals z throughout and conclude that $z_{n_k} \succsim z$ for all k. We can then set $z^* = z$ and we have $x_{n_k} \succsim z_{n_k} \succsim z = z^*$. Thus we found an element z^* and a subsequence (n_k) such that $x_{n_k} \succsim z^*$ with z^* being either z or one of z_n.

We now limit attention to the subsequence (n_k) and repeat the argument for (w_n). In a symmetric fashion, we now have a sub-subsequence (n_{k_l}) and w^* which is either w or one of w_{n_k} such that $w^* \succsim w_{n_{k_l}} \succsim y_{n_{k_l}}$. Importantly, whether $z^* = z_n$ or $z^* = z$, whether $w^* = w_{n_k}$ or $w^* = w$, we have $z^* \succ w^*$ (where this follows either from $z \succ w$, which was given, or from the claims proven above for the other three possibilities). Thus A3 can be used to derive the conclusion $x \succ y$.

Next assume that $z_n \to z$ and $w_n \to w$ are comparable but that they do not satisfy Condition A. This means that the satisfy Condition B, that is, that $(z_n) \subset X^i, z \in X^j$ and $(w_n) \subset X^i, w \in X^j$ for $i, j \in \{0,1\}$. But this also means that $i \neq j$ (or else Condition A would also hold). Further, because X^0 is closed, we have to have $(z_n), (w_n) \subset X^1$ while $z, w \in X^0$. As X^0 is convex, hence connected, we have $z' \in X^0$ such that $z \succ z^{'} \succ w$ (otherwise, we could use Lemma 6, applied to $z_n \to z$ and $w_n = w$ to get $w \succsim z$). Repeating the argument for the pair $z^{'} \succ w$, we conclude that there is also $w' \in X^0$ such that

$$z \succ z^{'} \succ w' \succ w.$$

Next we select elements $(z'_n), (w'_n) \subset X^1$ such that $z'_n \to z'$ and $w'_n \to w'$. Notice that this is possible as X^0 is a non-trivial subspace of X. Thus we have four sequences, $z_n \to z$, $w_n = w$, $z'_n \to z'$, $w'_n \to w'$ and two of which are comparable. Applying Lemma 8 consecutively, we conclude that there exists an $N > 1$ such that, for all $n, k, l, m > N$ we have

$$z_n \succ z'_k \succ w'_l \succ w_m.$$

Fix $k, l > N$ and set $z^* = z'_k$, $w^* = w'_l$. Thus, $z_n \succ z^* \succ w^* \succ w_n$ for all $n > N$. As we also have $x_n \succcurlyeq z_n$ and $w_n \succcurlyeq y_n$ for all n, we conclude that $x_n \succ z^* \succ w^* \succ y_n$ and apply A3 to conclude that $x \succ y$. □

We now turn to define the extension. Let there be given a bounded and continuous function $u : X^1 \to \mathbb{R}$ that represents $\succsim$ on X^1. We first note that

Lemma 11 Assume that $(x_n) \subset X^1$ is such that $x_n \to y \in X^0$. Then $\exists \lim_{n\to\infty} u(x_n)$.

Proof: Assume that $x_n \to y \in X^0$. We claim that there exists $a \in \mathbb{R}$ such that $u(x_n) \to a$. If $u(x_n) \to \sup_{x\in X^1} u(x)$ or $u(x_n) \to \inf_{x\in X^1} u(x)$ then $u(x_n)$ is convergent and we are done. Assume, then, that this is not the case. As u is bounded, we can find a number $a \in (\inf_{x\in X^1} u(x), \sup_{x\in X^1} u(x))$ and a subsequence $(x_{n_k})_k$ such that $u(x_{n_k}) \to_{k\to\infty} a$. If we also have $u(x_n) \to_{n\to\infty} a$, we are done. Otherwise, there exists $\varepsilon > 0$ such that, for infinitely many n's, $u(x_n) > a + \varepsilon$, or that, for infinitely many n's, $u(x_n) < a - \varepsilon$ (or both). This means that there is another subsequence $(x_{n_l})_l$ such that $u(x_{n_l}) \to_{l\to\infty} b$ with $|a - b| \geq \varepsilon$. Assume w.l.o.g. that $b \geq a + \varepsilon$. As u is continuous on X^1, and the latter is convex (and connected), we have points $z, w \in X^1$ such that $b - \frac{\varepsilon}{3} > u(z) > u(w) > a + \frac{\varepsilon}{3}$. But this means that, for large enough k, l, we have $x_{n_l} \succ z \succ w \succ x_{n_k}$ with $x_{n_k} \to_{k\to\infty} y$ and $x_{n_l} \to_{l\to\infty} y$. By A3 we should get $y \succ y$, a contradiction. Thus $u(x_n)$ is convergent. □

Lemma 12 For every $y \in X^0$ there exists $a \in \mathbb{R}$ such that, for every $(x_n) \subset X^1$ with $x_n \to y$, we have $\exists \lim_{n\to\infty} u(x_n) = a$.

Proof: Lemma 11 already established that every convergent sequence $x_n \to y \in X^0$ generates a convergent sequence of utilities. Clearly, this means that the limit is independent of the sequence. Explicitly, if $(x_n), (x'_n) \subset X^1$ are such that $x_n \to y \in X^0$ and $x'_n \to y$, we know that for some $a, a' \in \mathbb{R}$ we have $u(x_n) \to a$ and $u(x'_n) \to a'$. But if $a \neq a'$, we can generate a combined sequence whose utility has no limit. (Say, for $z_{2n} = x_n, z_{2n+1} = x'_n$, we get $z_n \to y$ but $u(z_n)$ is not convergent.) □

We can finally define the extension of u. For every $y \in X^0$ there exist sequences $(x_n) \subset X^1$ with $x_n \to y$. By Lemma 11 we have $\exists \lim_{n\to\infty} u(x_n)$ and by Lemma 12 its value is independent of the choice of the convergent sequence. Thus, setting

$$u(y) = \lim_{n\to\infty} u(x_n)$$

is well-defined. Observe that this is the unique extension of u to X^0 that holds a promise of continuity.

Lemma 13 u is continuous (also) on X^0.

Proof: Let there be given $y \in X^0$ and a convergent sequence $x_n \to y$. We need to show that $u(x_n) \to u(y)$. We will consider two special cases: $(x_n) \subset X^1$ and $(x_n) \subset X^0$. If we show that for each of these the conclusion $u(x_n) \to u(y)$ holds, we are done, as any other sequence can be split into two subsequences, one in X^0 and the other in X^1, and each of these, if infinite, has to yield u values that converge to $u(y)$.

When we consider $(x_n) \subset X^1$ we are back to the first part of the proof, where we showed that $u(x_n)$ is convergent, and that its limit has to be $u(y)$. Consider then a sequence $(x_n) \subset X^0$ such that $x_n \to y$ and assume that $u(x_n) \to u(y)$ doesn't hold. Then there exists $\varepsilon > 0$ such that, for infinitely many n's, $u(x_n) > u(y) + \varepsilon$, or that, for infinitely many n's, $u(x_n) < u(y) - \varepsilon$ (or both). For each n select a sequence $(x_n^k)_k \subset X^1$ such that $x_n^k \to_{k\to\infty} x_n$. For every m, pick n such that $\|x_n - y\| < \frac{1}{2m}$ and k such that $\|x_n^k - x_n\| < \frac{1}{2m}$ so that $(x_n^n) \subset X^1$ and $x_n^n \to_{n\to\infty} y$. However, $|u(x_n^n) - u(y)| \geq \varepsilon$, a contradiction. We thus conclude that u is continuous on X^0. □

Next, we wish to show that the continuous extension we constructed represents $\succsim$ also on its extended domain, X^0. We do this in two steps. First, we observe the following:

Lemma 14 For all $x, y \in X^0$, if $u(x) > u(y)$ then $x \succ y$.

Proof: By definition of u, we can take sequences $(x_n), (y_n) \subset X^1$ such that $x_n \to x$ and $y_n \to y$. Letting $\varepsilon = u(x) - u(y) > 0$ choose N large enough so that for all $n \geq N$ we have $|u(x_n) - u(x)|, |u(y_n) - u(y)| < \varepsilon/3$. As u is continuous on X^1 we can also find $z^*, w^* \in X^1$ so that $u(z^*) = u(x) - \varepsilon/3$; $u(w^*) = u(y) + \varepsilon/3$. Thus $u(x_n) > u(z^*) > u(w^*) > u(y_n)$ for all $n \geq N$. A3 implies that $x \succ y$. □

The next and final step of the proof is to show the converse, namely:

Lemma 15 For all $x, y \in X^0$, if $u(x) = u(y)$ then $x \sim y$.

Proof: We first prove an auxiliary claim:

Claim 1 *Assume that, for* $z, w \in X^0$*,* $u(z) = u(w) = a$ *but* $z \succ w$*. Let* $(z_n), (w_n) \subset X^1$ *converge to* z *and* w *respectively. Then* $\exists N$ *such that,* $\forall n \geq N$ *we have (i)* $u(z_n) \geq a$ *and (ii)* $u(w_n) \leq a$*.*

Proof of Claim: Suppose first that $u(z_n) < a$ occurs infinitely often. Let (n_k) be a sequence such that $u(z_{n_k}) < a$. Because $u(w_n) \to a$, for each such k we can find $m(n_k)$ such that $u\left(w_{m(n_k)}\right) > u(z_{n_k})$ and $m(n_k)$ increases in k. Thus we have two sequences $(z_{n_k}), \left(w_{m(n_k)}\right) \subset X^1$, converging to z and w, respectively, with $w_{m(n_k)} \succ z_{n_k}$. By A2, we get $w \succsim z$, a contradiction. By a similar argument, if $u(w_n) > a$ occurs infinitely often, we select such a subsequence $u(w_{n_k}) > a$ and $u\left(z_{m(n_k)}\right) < u(w_{n_k})$ and $w \succsim z$ follows again. Thus, $\exists N$ such that, $\forall n \geq N$ we have both $u(z_n) \geq a$ and $u(w_n) \leq a$. □

Equipped with this Claim we turn to prove the lemma. Assume that $x, y \in X^0$ satisfy $u(x) = u(y)$ but $x \succ y$. Because X^0 is connected and $\succsim$ satisfies A2, we have to have $z \in X^0$ such that $x \succ z \succ y$. Applying the same reasoning to z and y we can also get $w \in X^0$ such that $x \succ z \succ w \succ y$.

Let $a = u(x) = u(y)$. Applying Lemma 14, we know that $x \succ z \succ w \succ y$ and, indeed, $x \succsim z \succsim w \succsim y$ implies $u(x) \geq u(z) \geq u(w) \geq u(y)$ and thus we have $u(x) = u(z) = u(w) = u(y) = a$.

Let there be sequences $(x_n), (z_n), (w_n), (y_n) \subset X^1$ converging to x, z, w, y, respectively. Applying the Claim to $x \succ z$, we conclude that, from some N_1 on, $u(z_n) \leq a$. Applying the same Claim to $w \succ y$, we find that, from some N_2 on, $u(w_n) \geq a$. However, when we apply it to $z \succ w$ we find that, from some N_3 on, $u(z_n) \geq a$ and $u(w_n) \leq a$. For $n \geq \max(N_1, N_2, N_3)$ we have $u(z_n) = u(w_n) = a$. This means that $z_n \sim w_n$ and A2 yields $z \sim w$, a contradiction. □ ■

7.2 Examples

We use two continuity axioms, A2 and A3. A2 seems to be rather strong, and, as mentioned above, if we drop the comparability restriction, it is, per se,[9] stronger than the standard continuity assumption of consumer theory. Moreover, if we drop the comparability restriction, the two axioms are equivalent

[9]That is, without A1 necessarily assumed.

(for a weak order). Specifically, if we define

A2*. Universal Weak Preference Continuity: For all sequences $x_n \to x$ and $y_n \to y$, if $x_n \succsim y_n$ for all n, then $x \succsim y$.

A3*. Universal Strict Preference Continuity: For all sequences $x_n \to x$ and $y_n \to y$, and all $z, w \in X$, if $x_n \succsim z \succ w \succsim y_n$ for all n, then $x \succ y$.

We can state

Observation 1 *If $\succsim$ is a weak order on X, then A2* and A3* are equivalent.*

Proof: Assume first that $\succsim$ satisfies A2*. Then for the bundles in A3* we have $x \succsim z$ and $w \succsim y$, which implies $x \succ y$ by transitivity.

Next, assume that $\succsim$ satisfies A3*. We first claim that, for all sequences $x_n \to x$ and $y_n \to y$, if $x \succ y$, then there exists an N such that $x \succ y_n$ and $x_n \succ y$ for all $n > N$. To see this, suppose that the contrary holds. If $y_n \succsim x$ for infinitely many n's, then for these n's we have $y_n \succsim x \succ y \succsim y$, which by A3* implies $y \succ y$, a contradiction. Alternatively, $y \succsim x_n$ for infinitely many n's would imply $x \succsim x \succ y \succsim x_n$ and $x \succ x$.

To see that A2* holds, let there be given sequences $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n$ for all n, and assume that, contrary to our claim, $y \succ x$. For all n large enough, $y \succ x_n \succsim y_n \succ x$ by the argument above. Fix such a k so that $y \succ x_k \succsim y_k \succ x$. Apply the argument again to conclude that, for some N, we have $y_n \succ y_k$ for all $n > N$. Since $x_n \succsim y_n$ for all n, we have by transitivity $x_n \succ y_k$ for all $n > N$. So we have $x_n \succ y_k \succ x \succsim x$ for all $n > N$, which by A3* implies $x \succ x$, an impossibility. □

In light of this equivalence of the "universal" versions of the axioms (applying to all sequences, rather than only to comparable ones), one may wonder whether A3 is also needed, and, if so, maybe A3 can be assumed but A2 can be dispensed with. In the following we provide a few examples that show that none of the axioms is redundant. In the first five examples we have $n = 2$, $X = [0, 10]^2$ and $d = (1, 0)$, so that the principle is satisfied on the x_2 axis (X^0 consists of all the points with $x_1 = 0$) but not off the axis (X^0 consists of all the points with $x_1 > 0$). We define $\succsim$ by a numerical function v so that A1 is satisfied in all examples.

7.2.1 Example 1: A2 without A3 (I)

Let v be given by[10]:

$$v(x_1, x_2) = \begin{cases} 3 & x_1 = 0 \\ \sin\left(\frac{1}{x_1}\right) & x_1 > 0 \end{cases}$$

So the x_2 axis ($x_1 = 0$) is an indifference class that is preferred to anything else. Preference off the axis depend only on x_1, in a continuous way on the interior ($x_1 > 0$), but in a way that has no limit as we approach $x_1 = 0$.

To see that A2 is satisfied, consider $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ as in the antecedents of A2. Then if $x, y \in X^0$, the consequent $x \succsim y$ follows as $x \sim y$ for any $x, y \in X^0$. And if $x, y \in X^1$, then from some point on $x_n, y_n \in X^1$ and the consequent follows from the continuity of v on X^1. However, A3 isn't satisfied. More specifically, the claim of Lemma 9, which is an implication of A3, does not hold. To see this, define $x_n = \left(\frac{1}{(2n+\frac{1}{2})\pi}, 1\right)$; $y_n = \left(\frac{1}{(2n+\frac{3}{2})\pi}, 1\right)$ and $x = (0, 1)$ so that $x_n, y_n \to x$. Let $z = \left(\frac{2}{\pi}, 1\right)$ and $w = \left(\frac{2}{3\pi}, 1\right)$ so that $v(x_n) = v(z) = 1$ and $v(y_n) = v(w) = -1$. Thus, $x_n \succsim z$ and $w \succsim y_n$ but $w \succsim z$ doesn't hold. □

7.2.2 Example 2: A2 without A3 (II)

The previous example relies on the absence of a limit – preferences on X^1 have no "Cauchy sequences". The next example shows that this is only one problem that may arise, and that A3 may not hold even if preferences are very well-behaved on each of X^0, X^1. Let v be given by:

$$v(x_1, x_2) = \begin{cases} x_2 & x_1 = 0 \\ x_2 - 3 & x_1 > 0, x_2 < 5 \\ x_2 - 2 & x_1 > 0, x_2 = 5 \\ x_2 - 1 & x_1 > 0, x_2 > 5 \end{cases}$$

In the subspace $x_1 > 0$, $\succsim$ could also be represented by $v'(x_1, x_2) = x_2 - 2$ and it is clearly continuous there. But v is defined by taking $v'(x_1, x_2) = 3$

[10] Here and in the sequel we drop one set of parentheses for clarity. That is, $u_d((x_1, x_2))$ is denoted $u_d(x_1, x_2)$.

(corresponding to $x_2 = 5$) as a watershed, shifting the region $v'(x_1, x_2) > 3$ (corresponding to $x_2 > 5$) up by 1 and the region $v'(x_1, x_2) < 3$ (corresponding to $x_2 < 5$) down by 1. This generates "holes" in the range of U that could be skipped if we only had to worry about $x_1 > 0$. Yet, we cannot re-define U on this range to be continuous because we have points on the x_2 axis ($x_1 = 0$) that are in between preference-wise.

To see that A2 is satisfied, consider $x_n \to x$ and $y_n \to y$ with $x_n \gtrsim y_n$ as in the antecedents of A2. Then if $x, y \in X^0$, the consequent $x \gtrsim y$ follows because v is obviously continuous on X^0. And if $x, y \in X^1$, then from some point on $x_n, y_n \in X^1$ and the consequent follows from the fact that on X^1 the relation $\gtrsim$ could also be represented by v' which is continuous on X^1. However, A3 is violated. To see this, let $x_n = \left(1, 5 + \frac{1}{n}\right)$ and $y_n = \left(1, 5 - \frac{1}{n}\right)$ with $x = (1, 5)$ being their common limit. Take $z = (0, 4)$ and $w = (0, 3)$ so that $x_n \gtrsim z$ and $w \gtrsim y_n$ because $v\left(1, 5 + \frac{1}{n}\right) = 4 + \frac{1}{n} > v(0, 4)$ and $v(0, 3) = 3 > 2 + \frac{1}{n} = v\left(1, 5 - \frac{1}{n}\right)$. However, $w \gtrsim z$ doesn't hold. Thus, the claim of Lemma 9 is again violated. □

7.2.3 Example 3: Lemma 9

The next example satisfies the conclusion of Lemma 9 but not the other properties. Let v be defined by:

$$v(x_1, x_2) = \begin{cases} x_2 & x_1 = 0 \\ x_2 - 1 & x_1 > 0, x_2 < 5 \\ 9 - x_2 & x_1 > 0, x_2 \geq 5 \end{cases}$$

That is, as long as $x_2 \leq 5$ preferences are monotone in x_2 with a "jump" at the x_2 axis. However, when x_2 is above 5, the direction of preferences in the interior ($x_1 > 0$) reverses, but not on the axis.

These preferences do not satisfy A2. For example, let $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 6\right)$ with $x = (0, 4)$ and $y = (0, 6)$. Then we have $v(x_n) = v(y_n) = 3$ and thus $x_n \gtrsim y_n$, but $v(x) = 4 < 6 = v(y)$ so that $x \gtrsim y$ fails to hold.

At the same time, the conclusion of Lemma 9 holds. To see this, let $x_n \to x$ and $y_n \to x$. As v is uniformly continuous both on X^0 and on X^1, $\lim v(x_n)$ and $\lim v(y_n)$ exist and they are equal. This means that there can be no

$a = v(z)$ and $b = v(w)$ such that $v(x_n) \geq a > b \geq v(y_n)$ for all n, and if $x_n \succsim z$ and $w \succsim y_n$ for all n, $w \succsim z$ has to follow.

Finally, these preferences also do not satisfy A3. To see this, we can take $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 7\right)$ so that $v(x_n) = 3$ and $v(y_n) = 2$. For $z = (0,3)$ and $w = (0,2)$ we have $v(z) = 3, v(w) = 2$ so that $x_n \succsim z \succ w \succsim y_n$. But the limit points, $x = (0,4)$ and $y = (0,7)$ do not satisfy $x \succ y$ (in fact, the converse holds, that is, $y \succ x$). □

7.2.4 Example 4: A2 and Lemma 9 without A3

Next consider v defined by :

$$v(x_1, x_2) = \begin{cases} x_2 - 2 & x_1 > 0 \\ x_2 & x_1 = 0, \quad x_2 < 4 \\ 4 & x_1 = 0, \quad 4 \leq x_2 \leq 5 \\ x_2 - 1 & x_1 = 0, \quad x_2 > 5 \end{cases}$$

Thus, along the axis $x_1 = 0$, preferences are represented by a non-decreasing continuous function of x_2 that is constant on a given interval, and off it ($x_1 > 0$) they could also be represented by x_2.

We claim that these preferences satisfy A2 and the conclusion of Lemma 9 but not A3. Starting with A2, consider $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ as in the antecedents of A2. Then if $x_n, y_n \in X^0$, the consequent $x \succsim y$ follows because v is continuous on X^0. And if $x_n, y_n, x, y \in X^1$, the consequent follows from the fact that on X^1 the relation $\succsim$ could also be represented by $v' = x_2$. We are left with the interesting case in which $x_n, y_n \in X^1$ but $x, y \in X^0$. Because $x_n \succsim y_n$, we know that the second component of x_n is at least as high as is that of y_n, and it follows that the same inequality holds in the limit and $x \succsim y$.

The conclusion of Lemma 9 also holds because v is uniformly continuous on each of X^0 and X^1. Thus, $x_n \to x$ and $y_n \to x$ imply that $\lim v(x_n) = \lim v(y_n)$ (and that both exist).

However, A3 fails to hold. To see this, consider $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 5\right)$ with $x = (0,4)$ and $y = (0,5)$. For $z = (0,3)$ and $w = (0,2)$ we have $U(z) = 3, U(w) = 2$ so that $y_n \succsim z \succ w \succsim x_n$. But for limit points $x \sim y$, in violation of the axiom. □

7.2.5 Example 5: A3 without A2

Finally, we show that A3 does not imply A2. Let

$$v(x_1, x_2) = \begin{cases} -1 & x_1 > 0 \\ x_2 & x_1 = 0 \end{cases}$$

That is, the entire X^1 is a single indifference class that is below, preference-wise, the entire x_2 axis. We claim that these preferences satisfy A3 but not A2.

To see that A3 holds, consider $(x_n), (y_n)$ and x, y, z, w in X such that $x_n \to x$ and $y_n \to y$ and $x_n \succsim z \succ w \succsim y_n$. If $(x_n), (y_n) \subset X^0$ then we have $x, y \in X^0$. Because v is simply x_2 on X^0, the conclusion follows. If $(x_n), (y_n) \subset X^1$ we cannot have $x_n \succsim z \succ w \succsim y_n$ because $x_n \sim y_n$. Thus, A3 holds.

However, A2 can easily seen to be violated. For example, $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 5\right)$ satisfy $x_n \succsim y_n$ but at the limit we get $(0,5) \succ (0,4)$. □

7.2.6 Example 6: The Role of Connectedness

The following example shows that for Theorem 2 to hold, the set X has to be connected. Let

$$X = \left\{ (x_1, x_2) \;\middle|\; \begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ or \\ 2 \le x_2 \le 3 \end{array} \right\}$$

and define the following two functions on X:

$$u(x_1, x_2) = \begin{cases} -x_1 & 0 \le x_2 \le 1 \\ x_1 & 2 \le x_2 \le 3 \end{cases}$$

$$v(x_1, x_2) = \begin{cases} -x_1 & 0 \le x_2 \le 1 \\ x_1 + 1 & 2 \le x_2 \le 3 \end{cases}$$

Define $\succsim$ on X by maximization of v. As v is continuous, $\succsim$ satisfies axioms A1-A3. Note that u, restricted to $X^1 = \{(x_1, x_2) \,|\, x_1 > 0\}$, represents $\succsim$ as well. Indeed, it has a continuous extension to $X - u$ itself. However, it does not represent $\succsim$ on X^0, as u is constant on X^0 which isn't an equivalence class of $\succsim$ (say, $(2,0) \succ (1,0)$). □